Chapter
Complex Numbers and Quadratic Equations

Topic-1: Integral Powers of Iota, Algebraic Operations of Complex Numbers, Conjugate, Modulus and Argument or Amplitude of a Complex number

1 MCQs with One Correct Answer

1. If $\frac{w-\bar{w}z}{1-z}$ is purely real where $w=\alpha+i\beta, \beta \neq 0$ and $z \neq 1$, then the set of the values of z is **[2006 - 3M, –1]**
(a) $\{z : |z| = 1\}$ (b) $\{z : z = \bar{z}\}$
(c) $\{z : z \neq 1\}$ (d) $\{z : |z| = 1, z \neq 1\}$

2. For all complex numbers z_1, z_2 satisfying $|z_1|=12$ and $|z_2-3-4i|=5$, the minimum value of $|z_1-z_2|$ is **[2002S]**
(a) 0 (b) 2 (c) 7 (d) 17

3. If z_1, z_2 and z_3 are complex numbers such that **[2000S]**
$|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right| = 1$, then $|z_1+z_2+z_3|$ is
(a) equal to 1 (b) less than 1
(c) greater than 3 (d) equal to 3

4. If $\arg(z) < 0$, then $\arg(-z) - \arg(z) =$ **[2000S]**
(a) π (b) $-\pi$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

5. For positive integers n_1, n_2 the value of the expression $(1+i)^{n_1}+(1+i^3)^{n_1}+(1+i^5)^{n_2}+(1+i^7)^{n_2}$, where $i=\sqrt{-1}$ is a real number if and only if **[1996 - 1 Marks]**
(a) $n_1 = n_2 + 1$ (b) $n_1 = n_2 - 1$
(c) $n_1 = n_2$ (d) $n_1 > 0, n_2 > 0$

6. Let z and ω be two complex numbers such that $|z| \leq 1$, $|\omega| \leq 1$ and $|z+i\omega| = |z-i\bar{\omega}| = 2$ then z equals **[1995S]**
(a) 1 or i (b) i or $-i$
(c) 1 or -1 (d) i or -1

7. Let z and ω be two non zero complex numbers such that $|z| = |\omega|$ and $\text{Arg}\, z + \text{Arg}\, \omega = \pi$, then z equals **[1995S]**
(a) ω (b) $-\omega$ (c) $\bar{\omega}$ (d) $-\bar{\omega}$

8. The smallest positive integer n for which **[1980]**
$\left(\frac{1+i}{1-i}\right)^n = 1$ is
(a) $n=8$ (b) $n=16$
(c) $n=12$ (d) none of these

2 Integer Value Answer/ Non-Negative Integer

9. Let $A = \left\{\frac{1967+1686\, i \sin\theta}{7-3\, i \cos\theta} : \theta \in \mathbb{R}\right\}$. If A contains exactly one positive integer n, then the value of n is **[Adv. 2023]**

10. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)$, where $i=\sqrt{-1}$. The value of the expression $\frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|}$ is **[Adv. 2015]**

11. If z is any complex number satisfying $|z-3-2i| \leq 2$, then the minimum value of $|2z-6+5i|$ is **[2011]**

3 Numeric/ New Stem Based Questions

12. Let z be a complex number with non-zero imaginary part. If $\frac{2+3z+4z^2}{2-3z+4z^2}$ is a real number, then the value of $|z|^2$ is ________. **[Adv. 2022]**

13. Let $\bar{z}$ denote the complex conjugate of a complex number z and let $i=\sqrt{-1}$. In the set of complex numbers, the number of distinct roots of the equation $\bar{z}-z^2 = i(\bar{z}+z^2)$ is ________. **[Adv. 2022]**

4 Fill in the Blanks

14. If the expression [1987 - 2 Marks]

$$\frac{\left[\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)+i\tan(x)\right]}{\left[1+2i\sin\left(\frac{x}{2}\right)\right]}$$

is real, then the set of all possible values of x is ....

5 True / False

15. For complex number $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$, if $x_1 \le x_2$ and $y_1 \le y_2$. Then for all complex numbers z with $1 \cap z$, we have $\frac{1-z}{1+z} \cap 0$.

[1981 - 2 Marks]

6 MCQs with One or More than One Correct Answer

16. Let $S=\{a+b\sqrt{2}:a,b\in\mathbb{Z}\}, T_1=\{(-1+\sqrt{2})^n:n\in\mathbb{N}\}$ and $T_2=\{(1+\sqrt{2})^n:n\in\mathbb{N}\}$. Then which of the following statements is (are) TRUE? **[Adv. 2024]**

(a) $\mathbb{Z}\cup T_1\cup T_2\subset S$

(b) $T_1\cap\left(0,\frac{1}{2024}\right)=\phi$, where ϕ denotes the empty set.

(c) $T_2\cap(2024,\infty)\ne\phi$

(d) For any given $a, b\in\mathbb{Z}$, $\cos(\pi(a+b\sqrt{2}))+i\sin(\pi(a+b\sqrt{2}))\in\mathbb{Z}$ if and only if $b=0$, where $i=\sqrt{-1}$.

17. Let $\bar{z}$ denote the complex conjugate of a complex number z. If z is a non-zero complex number for which both real and imaginary parts of $(\bar{z})^2+\frac{1}{z^2}$ are integers, then which of the following is/are possible value(s) of $|z|$? **[Adv. 2022]**

(a) $\left(\frac{43+3\sqrt{205}}{2}\right)^{\frac{1}{4}}$ (b) $\left(\frac{7+\sqrt{33}}{4}\right)^{\frac{1}{4}}$

(c) $\left(\frac{9+\sqrt{65}}{4}\right)^{\frac{1}{4}}$ (d) $\left(\frac{7+\sqrt{13}}{6}\right)^{\frac{1}{4}}$

18. Let S be the set of all complex numbers z satisfying $|z^2+z+1|=1$. Then which of the following statements is/are TRUE? **[Adv. 2020]**

(a) $\left|z+\frac{1}{2}\right|\le\frac{1}{2}$ for all $z\in S$

(b) $|z|\le 2$ for all $z\in S$

(c) $\left|z+\frac{1}{2}\right|\ge\frac{1}{2}$ for all $z\in S$

(d) The set S has exactly four elements

19. Let s, t, r be non-zero complex numbers and L be the set of solutions $z=x+iy$ $(x, y, \in\mathbb{R}, i=\sqrt{-1})$ of the equation $sz+t\bar{z}+r=0$, where $\bar{z}=x-iy$. Then, which of the following statement(s) is (are) TRUE? **[Adv. 2018]**

(a) If L has exactly one element, then $|s|\ne|t|$

(b) If $|s|=|t|$, then L has infinitely many elements

(c) The number of elements in $L\cap\{z:|z-1+i|=5\}$ is at most 2

(d) If L has more than one element, then L has infinitely many elements

20. For a non-zero complex number z, let arg(z) denote the principal argument with $-\pi<\arg(z)\le\pi$. Then, which of the following statement (s) is (are) **FALSE?** **[Adv. 2018]**

(a) $\arg(-1-i)=\frac{\pi}{4}$, where $i=\sqrt{-1}$

(b) The function $f:\mathbb{R}\to(-\pi,\pi]$, defined by $f(t)=\arg(-1+it)$ for all $t\in\mathbb{R}$, is continuous at all points of $\mathbb{R}$, where $i=\sqrt{-1}$

(c) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right)-\arg(z_1)+\arg(z_2)$ is an integer multiple of 2π

(d) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right)=\pi$, lies on a straight line

21. Let a, b, x and y be real numbers such that $a-b=1$ and $y\ne 0$. If the complex number $z=x+iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right)=y$, then which of the following is(are) possible value(s) of x? **[Adv. 2017]**

(a) $-1+\sqrt{1-y^2}$ (b) $-1-\sqrt{1-y^2}$

(c) $1+\sqrt{1+y^2}$ (d) $1-\sqrt{1+y^2}$

22. Let z_1 and z_2 be two distinct complex numbers and let $z=(1-t)z_1+tz_2$ for some real number t with $0<t<1$. If Arg (w) denotes the principal argument of a non-zero complex number w, then **[2010]**

(a) $|z-z_1|+|z-z_2|=|z_1-z_2|$

(b) Arg $(z-z_1)=$ Arg $(z-z_2)$

(c) $\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix}$

(d) Arg $(z-z_1)=$ Arg (z_2-z_1)

23. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then **[1998 - 2 Marks]**

(a) $x=3, y=2$ (b) $x=1, y=3$
(c) $x=0, y=3$ (d) $x=0, y=0$

24. The value of the sum $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals **[1998 - 2 Marks]**

(a) i (b) $i-1$ (c) $-i$ (d) 0

25. The value of $\sum_{k=1}^{6} (\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7})$ is **[1987 - 2 Marks]**

(a) -1 (b) 0 (c) $-i$ (d) i
(e) None

26. If z_1 and z_2 are two nonzero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then Arg z_1 – Arg z_2 is equal to **[1987 - 2 Marks]**

(a) $-\pi$ (b) $-\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{2}$
(e) π

27. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{z_1 + z_2}{z_1 - z_2}$ may be **[1986 - 2 Marks]**

(a) zero (b) real and positive
(c) real and negative (d) purely imaginary
(e) none of these.

28. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and Re$(z_1 \bar{z}_2) = 0$, then the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies – **[1985 - 2 Marks]**

(a) $|w_1| = 1$ (b) $|w_2| = 1$
(c) Re$(w_1 \bar{w}_2) = 0$ (d) none of these

7 Match the Following

29. Let z be a complex number satisfying $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$, where $\bar{z}$ denotes the complex conjugate of z. Let the imaginary part of z be non-zero.

Match each entry in **List-I** to the correct entries in **List-II**.

List-I	List-II
(P) $|z|^2$ is equal to	(1) 12
(Q) $|z-\bar{z}|^2$ is equal to	(2) 4
(R) $|z|^2 + |z+\bar{z}|^2$ is equal to	(3) 8
(S) $|z+1|^2$ is equal to	(4) 10
	(5) 7

The correct option is: **[Adv. 2023]**

(a) (P) → (1), (Q) → (3), (R) → (5), (S) → (4)
(b) (P) → (2), (Q) → (1), (R) → (3), (S) → (5)
(c) (P) → (2), (Q) → (4), (R) → (5), (S) → (1)
(d) (P) → (2), (Q) → (3), (R) → (5), (S) → (4)

30. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; $k = 1, 2, ..., 9$. **[Adv. 2014]**

List-I	List-II
P. For each z_k there exists as z_j such that $z_k . z_j = 1$	**1.** True
Q. There exists a $k \in \{1, 2, ..., 9\}$ such that $z_1 . z = z_k$ has no solution z in the set of complex numbers	**2.** False
R. $\frac{|1-z_1||1-z_2|....|1-z_9|}{10}$ equals	**3.** 1
S. $1 - \sum_{k=1}^{9} \cos\left(\frac{2k\pi}{10}\right)$ equals	**4.** 2

	P	Q	R	S
(a)	1	2	4	3
(b)	2	1	3	4
(c)	1	2	3	4
(d)	2	1	4	3

4 Fill in the Blanks

14. If the expression [1987 - 2 Marks]

$$\frac{\left[\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)+i\tan(x)\right]}{\left[1+2\,i\sin\left(\frac{x}{2}\right)\right]}$$

is real, then the set of all possible values of x is ....

5 True / False

15. For complex number $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$, if $x_1 \le x_2$ and $y_1 \le y_2$. Then for all complex numbers z with $1 \cap z$, we have $\frac{1-z}{1+z} \cap 0$. **[1981 - 2 Marks]**

6 MCQs with One or More than One Correct Answer

16. Let $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}, T_1 = \{(-1+\sqrt{2})^n : n \in \mathbb{N}\}$ and $T_2 = \{(1+\sqrt{2})^n : n \in \mathbb{N}\}$. Then which of the following statements is (are) TRUE? **[Adv. 2024]**

(a) $\mathbb{Z} \cup T_1 \cup T_2 \subset S$

(b) $T_1 \cap \left(0, \frac{1}{2024}\right) = \phi$, where ϕ denotes the empty set.

(c) $T_2 \cap (2024, \infty) \ne \phi$

(d) For any given $a, b \in \mathbb{Z}$, $\cos(\pi(a + b\sqrt{2})) + i \sin(\pi(a + b\sqrt{2})) \in \mathbb{Z}$ if and only if $b = 0$, where $i = \sqrt{-1}$.

17. Let $\bar{z}$ denote the complex conjugate of a complex number z. If z is a non-zero complex number for which both real and imaginary parts of $(\bar{z})^2 + \frac{1}{z^2}$ are integers, then which of the following is/are possible value(s) of $|z|$? **[Adv. 2022]**

(a) $\left(\frac{43+3\sqrt{205}}{2}\right)^{\frac{1}{4}}$ (b) $\left(\frac{7+\sqrt{33}}{4}\right)^{\frac{1}{4}}$

(c) $\left(\frac{9+\sqrt{65}}{4}\right)^{\frac{1}{4}}$ (d) $\left(\frac{7+\sqrt{13}}{6}\right)^{\frac{1}{4}}$

18. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE? **[Adv. 2020]**

(a) $\left|z + \frac{1}{2}\right| \le \frac{1}{2}$ for all $z \in S$

(b) $|z| \le 2$ for all $z \in S$

(c) $\left|z + \frac{1}{2}\right| \ge \frac{1}{2}$ for all $z \in S$

(d) The set S has exactly four elements

19. Let s, t, r be non-zero complex numbers and L be the set of solutions $z = x + iy$ $(x, y, \in \mathbb{R}, i = \sqrt{-1})$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE? **[Adv. 2018]**

(a) If L has exactly one element, then $|s| \ne |t|$

(b) If $|s| = |t|$, then L has infinitely many elements

(c) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(d) If L has more than one element, then L has infinitely many elements

20. For a non-zero complex number z, let arg(z) denote the principal argument with $-\pi < \arg(z) \le \pi$. Then, which of the following statement (s) is (are) **FALSE?** **[Adv. 2018]**

(a) $\arg(-1-i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$

(b) The function $f : \mathbb{R} \to (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in \mathbb{R}$, is continuous at all points of $\mathbb{R}$, where $i = \sqrt{-1}$

(c) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(d) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line

21. Let a, b, x and y be real numbers such that a – b = 1 and y ≠ 0. If the complex number z = x + iy satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x ? **[Adv. 2017]**

(a) $-1+\sqrt{1-y^2}$ (b) $-1-\sqrt{1-y^2}$

(c) $1+\sqrt{1+y^2}$ (d) $1-\sqrt{1+y^2}$

22. Let z_1 and z_2 be two distinct complex numbers and let $z = (1-t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg (w) denotes the principal argument of a non-zero complex number w, then **[2010]**

(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(b) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$

(c) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix}$

(d) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

8 Comprehension/Passage Based Questions

PASSAGE-1

Let $S = S_1 \cap S_2 \cap S_3$, where

$S_1 = \{z \in \mathbb{C} : |z| < 4\}$, $S_2 = \left\{z \in \mathbb{C} : \text{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0\right\}$

and $S_3 = \{z \in \mathbb{C} : \text{Re}\, z > 0\}$. [Adv. 2013]

31. Area of $S =$

(a) $\frac{10\pi}{3}$ (b) $\frac{20\pi}{3}$ (c) $\frac{16\pi}{3}$ (d) $\frac{32\pi}{3}$

32. $\min_{z \in S} |1 - 3i - z| =$

(a) $\frac{2-\sqrt{3}}{2}$ (b) $\frac{2+\sqrt{3}}{2}$ (c) $\frac{3-\sqrt{3}}{2}$ (d) $\frac{3+\sqrt{3}}{2}$

PASSAGE-2

Let A, B, C be three sets of complex numbers as defined below

$A = \{z : \text{Im}\, z \geq 1\}$ [2008]

$B = \{z : |z - 2 - i| = 3\}$

$C = \{z : \text{Re}((1-i)z) = \sqrt{2}\}$

33. The number of elements in the set $A \cap B \cap C$ is

(a) 0 (b) 1 (c) 2 (d) ∞

34. Let z be any point in $A \cap B \cap C$.
Then, $|z+1-i|^2 + |z-5-i|^2$ lies between

(a) 25 and 29 (b) 30 and 34
(c) 35 and 39 (d) 40 and 44

35. Let z be any point $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between

(a) −6 and 3 (b) −3 and 6
(c) −6 and 6 (d) −3 and 9

10 Subjective Problems

36. If z_1 and z_2 are two complex numbers such taht $|z_1| < 1 < |z_2|$ then prove that $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$. [2003 - 2 Marks]

37. Let $Z_1 = 10 + 6i$ and $Z_2 = 4 + 6i$. If Z is any complex number such that the argument of $\frac{(Z - Z_1)}{(Z - Z_2)}$ is $\frac{\pi}{4}$, then prove that $|Z - 7 - 9i| = 3\sqrt{2}$. [1990 - 4 Marks]

38. Show that the area of the triangle on the Argand diagram formed by the complex numbers z, iz and $z + iz$ is $\frac{1}{2}|z|^2$. [1986 - 2½ Marks]

39. Find the real values of x and y for which the following equation is satisfied $\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$ [1980]

40. If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, prove that $(x^2 + y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$. [1979]

41. Express $\frac{1}{1 - \cos\theta + 2i\sin\theta}$ in the form $x + iy$. [1978]

Topic-2: Rotational Theorem, Square Root of a Complex Number, Cube Roots of Unity, Geometry of Complex Numbers, De-moiver's Theorem, Powers of Complex Numbers

1 MCQs with One Correct Answer

1. Let $\theta_1, \theta_2, ..., \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + ... + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}$, $z_k = z_{k-1}e^{i\theta_k}$ for $k = 2, 3, ..., 10$, where $i = \sqrt{-1}$. Consider the statements P and Q given below :

$P\; |z_2 - z_1| + |z_3 - z_2| + ... + |z_{10} - z_9| + |z_1 - z_{10}| \leq 2\pi$

$Q\; |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + ... + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \leq 4\pi$

[Adv. 2021]

(a) P is **TRUE** and Q is **FALSE**
(b) Q is **TRUE** and P is **FALSE**
(c) both P and Q are **TRUE**
(d) both P and Q are **FALSE**

2. Let S be the set of all complex numbers z satisfying $|z - 2 + i| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is [Adv. 2019]

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $-\frac{\pi}{2}$

3. Let complex numbers α and $\frac{1}{\alpha}$ lie on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$ [Adv. 2013]

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{7}}$ (d) $\frac{1}{3}$

4. Let z be a complex number such that the imaginary part of z is non-zero and $a = z^2 + z + 1$ is real. Then a cannot take the value **[2012]**

(a) -1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

5. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation : $z\bar{z}^3 + \bar{z}z^3 = 350$ is **[2009]**

(a) 48 (b) 32 (c) 40 (d) 80

6. Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is **[2009]**

(a) $\frac{1}{\sin 2°}$ (b) $\frac{1}{3\sin 2°}$ (c) $\frac{1}{2\sin 2°}$ (d) $\frac{1}{4\sin 2°}$

7. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by **[2008]**

(a) $6+7i$ (b) $-7+6i$ (c) $7+6i$ (d) $-6+7i$

8. If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\frac{z}{1-z^2}$ lie on

(a) a line not passing through the origin **[2007 -3 marks]**

(b) $|z| = \sqrt{2}$

(c) the x-axis

(d) the y-axis

9. A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. From there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. Then the position of P in the Argand plane is **[2007 -3 marks]**

(a) $3e^{i\pi/4} + 4i$ (b) $(3-4i)e^{i\pi/4}$

(c) $(4+3i)e^{i\pi/4}$ (d) $(3+4i)e^{i\pi/4}$

10. a, b, c are integers, not all simultaneously equal and ω is cube root of unity ($\omega \neq 1$), then minimum value of $|a + b\omega + c\omega^2|$ is **[2005S]**

(a) 0 (b) 1 (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

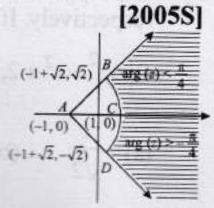
11. The locus of z which lies in shaded region (excluding the boundaries) is best represented by **[2005S]**

(a) $z : |z+1| > 2$ and $|\arg(z+1)| < \pi/4$

(b) $z : |z-1| > 2$ and $|\arg(z-1)| < \pi/4$

(c) $z : |z+1| < 2$ and $|\arg(z+1)| < \pi/2$

(d) $z : |z-1| < 2$ and $|\arg(z+1)| < \pi/2$



12. If $\omega (\neq 1)$ be a cube root of unity and $(1+\omega^2)^n = (1+\omega^4)^n$, then the least positive value of n is **[2004S]**

(a) 2 (b) 3 (c) 5 (d) 6

13. If $|z| = 1$ and $\omega = \frac{z-1}{z+1}$ (where $z \neq -1$), then Re(ω) is **[2003S]**

(a) 0 (b) $-\frac{1}{|z+1|^2}$

(c) $\left|\frac{z}{z+1}\right| \cdot \frac{1}{|z+1|^2}$ (d) $\frac{\sqrt{2}}{|z+1|^2}$

14. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, then the value of the det.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$$ is **[2002 - 2 Marks]**

(a) 3ω (b) $3\omega(\omega-1)$

(c) $3\omega^2$ (d) $3\omega(1-\omega)$

15. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1-i\sqrt{3}}{2}$ are the vertices of a triangle which is **[2001S]**

(a) of area zero (b) right-angled isosceles

(c) equilateral (d) obtuse-angled isosceles

16. Let z_1 and z_2 be n^{th} roots of unity which subtend a right angle at the origin. Then n must be of the form **[2001S]**

(a) $4k+1$ (b) $4k+2$

(c) $4k+3$ (d) $4k$

17. If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365}$ is equal to **[1999 - 2 Marks]**

(a) $1-i\sqrt{3}$ (b) $-1+i\sqrt{3}$

(c) $i\sqrt{3}$ (d) $-i\sqrt{3}$

18. If ω ($\neq 1$) is a cube root of unity and $(1+\omega)^7 = A + B\omega$ then A and B are respectively **[1995S]**

(a) 0, 1 (b) 1, 1 (c) 1, 0 (d) -1, 1

19. If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that $c = (1-r)a + rb$ and $w = (1-r)u + rv$, where r is a complex number, then the two triangles **[1985 - 2 Marks]**

(a) have the same area (b) are similar

(c) are congruent (d) none of these

20. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if **[1983 - 1 Mark]**

(a) $z_1 + z_4 = z_2 + z_3$ (b) $z_1 + z_3 = z_2 + z_4$

(c) $z_1 + z_2 = z_3 + z_4$ (d) None of these

21. If $z = x + iy$ and $\omega = (1 - iz)/(z - i)$, then $|\omega| = 1$ implies that, in the complex plane, **[1983 - 1 Mark]**
(a) z lies on the imaginary axis
(b) z lies on the real axis
(c) z lies on the unit circle
(d) None of these

22. The inequality $|z-4| < |z-2|$ represents the region given by **[1982 - 2 Marks]**
(a) $\text{Re}(z) \geq 0$ (b) $\text{Re}(z) < 0$
(c) $\text{Re}(z) > 0$ (d) none of these

23. If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then **[1982 - 2 Marks]**
(a) $\text{Re}(z) = 0$ (b) $\text{Im}(z) = 0$
(c) $\text{Re}(z) > 0, \text{Im}(z) > 0$ (d) $\text{Re}(z) > 0, \text{Im}(z) < 0$

24. The complex numbers $z = x + iy$ which satisfy the equation $\left|\frac{z - 5i}{z + 5i}\right| = 1$ lie on **[1981 - 2 Marks]**
(a) the x-axis
(b) the straight line $y = 5$
(c) a circle passing through the origin
(d) none of these

25. If the cube roots of unity are $1, \omega, \omega^2$, then the roots of the equation $(x-1)^3 + 8 = 0$ are **[1979]**
(a) $-1, 1+2\omega, 1+2\omega^2$ (b) $-1, 1-2\omega, 1-2\omega^2$
(c) $-1, -1, -1$ (d) None of these

2 Integer Value Answer/ Non-Negative Integer

26. For a complex number z, let $\text{Re}(z)$ denote the real part of z. Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $\text{Re}(z_1) > 0$ and $\text{Re}(z_2) < 0$, is _____ **[Adv. 2020]**

27. Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set $\left\{\left|a + b\omega + c\omega^2\right|^2 : a, b, c \text{ distinct non-zero integers}\right\}$ equals _____. **[Adv. 2019]**

4 Fill in the Blanks

28. The value of the expression $1\cdot(2-\omega)(2-\omega^2) + 2\cdot(3-\omega)(3-\omega^2) + ... + (n-1).(n-\omega)(n-\omega^2)$, where ω is an imaginary cube root of unity, is..... **[1996 - 2 Marks]**

29. Suppose Z_1, Z_2, Z_3 are the vertices of an equilateral triangle inscribed in the circle $|Z| = 2$. If $Z_1 = 1 + i\sqrt{3}$ then $Z_2 = ...$, $Z_3 = ...$ **[1994 - 2 Marks]**

30. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M represent the complex numbers $1 + i$ and $2 - i$ respectively, then A represents the complex number ....or......... **[1993 - 2 Marks]**

31. If a and b are the numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then $a = ...$ and $b = ...$ **[1989 - 2 Marks]**

32. For any two complex numbers z_1, z_2 and any real number a and b. **[1988 - 2 Marks]**
$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = ...$

5 True / False

33. The cube roots of unity when represented on Argand diagram form the vertices of an equilateral triangle. **[1988 - 1 Mark]**

34. If three complex numbers are in A.P. then they lie on a circle in the complex plane. **[1985 - 1 Mark]**

35. If the complex numbers, Z_1, Z_2 and Z_3 represent the vertices of an equilateral triangle such that $|Z_1| = |Z_2| = |Z_3|$ then $Z_1 + Z_2 + Z_3 = 0$. **[1984 - 1 Mark]**

6 MCQs with One or More than One Correct Answer

36. Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$.

Suppose $S = \left\{z \in C : Z = \frac{1}{a + ibt}, \in \mathbb{R}^+, t \neq 0\right\}$, where $i = \sqrt{-1}$. If $z = x + iy$ and $z \in S$, then (x, y) lies on **[JEE Adv. 2016]**
(a) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$
(b) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$
(c) the x-axis for $a \neq 0, b = 0$
(d) the y-axis for $a = 0, b \neq 0$

37. Let $w = \frac{\sqrt{3} + i}{2}$ and $P = \{w^n : n = 1, 2, 3, ...\}$. Further $H_1 = \left\{z \in \mathbb{C} : \text{Re}\, z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in \mathbb{C} : \text{Re}\, z < \frac{-1}{2}\right\}$, where c is the set of all complex numbers. If $z_1 \in P \cap H_1, z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1 O z_2 =$ **[Adv. 2013]**
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

38. If ω is an imaginary cube root of unity, then $(1+\omega-\omega^2)^7$ equals **[1998 - 2 Marks]**

(a) 128ω (b) -128ω (c) $128\omega^2$ (d) $-128\omega^2$

7 Match the Following

39. Match the statements in **Column I** with those in **Column II.** **[2010]**

[**Note :** Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z.]

Column I

(A) The set of points z satisfying $|z-i|z||=|z+i|z||$ is contained in or equal to

(B) The set of points z satisfying $|z+4|+|z-4|=10$ is contained in or equal to

(C) If $|w|=2$, then the set of points $z=w-\frac{1}{w}$ is contained in or equal to

(D) If $|w|=1$, then the set of points $z=w+\frac{1}{w}$ is contained in or equal to.

Column II

(p) an ellipse with eccentricity $\frac{4}{5}$

(q) the set of points z satisfying $\text{Im}\, z=0$

(r) the set of points z satisfying $|\text{Im}\, z|\le 1$

(s) the set of points z satisfying $|\text{Re}\, z|<2$

(t) the set of points z satisfying $|z|\le 3$

40. $z\ne 0$ is a complex number **[1992 - 2 Marks]**

Column I

(A) $\text{Re}\, z=0$

(B) $\text{Arg}\, z=\frac{\pi}{4}$

Column II

(p) $\text{Re}\, z^2=0$

(q) $\text{Im}\, z^2=0$

(r) $\text{Re}\, z^2=\text{Im}\, z^2$

10 Subjective Problems

41. If one the vertices of the square circumscribing the circle $|z-1|=\sqrt{2}$ is $2+\sqrt{3}\,i$. Find the other vertices of the square. **[2005 - 4 Marks]**

42. Find the centre and radius of circle given by

$$\left|\frac{z-\alpha}{z-\beta}\right|=k, k\ne 1$$

where, $z=x+iy$, $\alpha=\alpha_1+i\alpha_2$, $\beta=\beta_1+i\beta_2$ **[2004 - 2 Marks]**

43. Prove that there exists no complex number z such that $|z|<\frac{1}{3}$ and $\sum_{r=1}^{n}a_r z^r=1$ where $|a_r|<2$. **[2003 - 2 Marks]**

44. Let a complex number α, $\alpha\ne 1$, be a root of the equation $z^{p+q}-z^p-z^q+1=0$, where p, q are distinct primes. Show that either $1+\alpha+\alpha^2+....+\alpha^{p-1}=0$ or $1+\alpha+\alpha^2+...+\alpha^{q-1}=0$, but not both together. **[2002 - 5 Marks]**

45. For complex numbers z and w, prove that $|z|^2\omega-|\omega|^2 z=z-\omega$ if and only if $z=\omega$ or $z\bar{\omega}=1$. **[1999 - 10 Marks]**

46. Let z_1 and z_2 be roots of the equation $z^2+pz+q=0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB=\alpha\ne 0$ and $OA=OB$, where O is the origin, prove that

$$p^2=4q\cos^2\left(\frac{\alpha}{2}\right)$$ **[1997 - 5 Marks]**

47. Find all non-zero complex numbers Z satisfying $\overline{Z}=iZ^2$. **[1996 - 2 Marks]**

48. If $|Z|\le 1$, $|W|\le 1$, show that

$$|Z-W|^2\le(|Z|-|W|)^2+(Arg\,Z-Arg\,W)^2$$ **[1995 - 5 Marks]**

49. If $iz^3+z^2-z+i=0$, then show that $|z|=1$. **[1995 - 5 Marks]**

50. If $1, a_1, a_2 ......, a_{n-1}$ are the n roots of unity, then show that $(1-a_1)(1-a_2)(1-a_3)....(1-a_{n-1})=n$ **[1984 - 2 Marks]**

51. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1{}^2+z_2{}^2-z_1z_2=0$. **[1983 - 3 Marks]**

52. Let the complex number z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$. **[1981 - 4 Marks]**

53. If $x = a + b, y = a\gamma + b\beta$ and $z = a\beta + b\gamma$ where γ and β are the complex cube roots of unity, show that $xyz = a^3 + b^3$. **[1978]**

Topic-3: Solutions of Quadratic Equations, Sum and Product of Roots, Nature of Roots, Relation Between Roots and Co-efficients, Formation of an Equation with Given Roots

1 MCQs with One Correct Answer

1. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of **[Adv. 2020]**

$ac(a-c) + ad(a-d) + bc(b-c) + bd(b-d)$ is

(a) 0 (b) 8000
(c) 8080 (d) 16000

2. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose θ_1 and B_1 are the roots of the equation $x^2 - 2x \sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals **[Adv. 2016]**

(a) $2(\sec\theta - \tan\theta)$ (b) $2\sec\theta$
(c) $-2\tan\theta$ (d) 0

3. The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then the equation $p(p(x)) = 0$ has **[Adv. 2014]**

(a) one purely imaginary root
(b) all real roots
(c) two real and two purely imaginary roots
(d) neither real nor purely imaginary roots

4. If $a \in R$ and the equation

$-3(x-[x])^2 + 2(x-[x]) + a^2 = 0$

(where $[x]$ denotes the greatest integer $\leq x$) has no integral solution, then all possible values of a lie in the interval: **[Main 2014]**

(a) $(-2,-1)$ (b) $(-\infty,-2)\cup(2,\infty)$
(c) $(-1,0)\cup(0,1)$ (d) $(1,2)$

5. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is **[2011]**

(a) 1 (b) 2 (c) 3 (d) 4

6. Let (x_0, y_0) be the solution of the following equations

$(2x)^{\ell n 2} = (3y)^{\ell n 3}$; $3^{\ell n x} = 2^{\ell n y}$.

Then x_0 is **[2011]**

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 6

7. Let p and q be real numbers such that $p \neq 0, p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is **[2010]**

(a) $(p^3+q)x^2 - (p^3+2q)x + (p^3+q) = 0$
(b) $(p^3+q)x^2 - (p^3-2q)x + (p^3+q) = 0$
(c) $(p^3-q)x^2 - (5p^3-2q)x + (p^3-q) = 0$
(d) $(p^3-q)x^2 - (5p^3+2q)x + (p^3-q) = 0$

8. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is **[2007 -3 marks]**

(a) $\frac{2}{9}(p-q)(2q-p)$ (b) $\frac{2}{9}(q-p)(2p-q)$
(c) $\frac{2}{9}(q-2p)(2q-p)$ (d) $\frac{2}{9}(2p-q)(2q-p)$

9. Let a, b, c be the sides of a triangle where $a \neq b \neq c$ and $\lambda \in R$. If the roots of the equation $x^2 + 2(a+b+c)x + 3\lambda(ab+bc+ca) = 0$ are real, then **[2006 - 3M, –1]**

(a) $\lambda < \frac{4}{3}$ (b) $\lambda > \frac{5}{3}$
(c) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (d) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

10. If one root is square of the other root of the equation $x^2 + px + q = 0$, then the relation between p and q is **[2004S]**

(a) $p^3 - q(3p-1) + q^2 = 0$
(b) $p^3 - q(3p+1) + q^2 = 0$
(c) $p^3 + q(3p-1) + q^2 = 0$
(d) $p^3 + q(3p+1) + q^2 = 0$

11. For the equation $3x^2 + px + 3 = 0, p > 0$, if one of the root is square of the other, then p is equal to **[2000S]**

(a) 1/3 (b) 1 (c) 3 (d) 2/3

12. If $b > a$, then the equation $(x-a)(x-b)-1 = 0$ has **[2000S]**

(a) both roots in (a, b)
(b) both roots in $(-\infty, a)$
(c) both roots in $(b, +\infty)$
(d) one root in $(-\infty, a)$ and the other in $(b, +\infty)$

13. If α and β $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then **[2000S]**

(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$

14. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then **[1999 - 2 Marks]**

(a) $a < 2$ (b) $2 \le a \le 3$
(c) $3 < a \le 4$ (d) $a > 4$

15. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + 1 = 0$ having real roots is **[1994]**

(a) 15 (b) 9
(c) 7 (d) 8

16. Let α, β be the roots of the equation $(x-a)(x-b) = c$, $c \ne 0$. Then the roots of the equation $(x-\alpha)(x-\beta) + c = 0$ are **[1992 - 2 Marks]**

(a) a, c (b) b, c
(c) a, b (d) $a+c, b+c$

17. Let a, b, c be real numbers, $a \ne 0$. If α is a root of $a^2x^2 + bx + c = 0$. β is the root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies **[1989 - 2 Marks]**

(a) $\gamma = \frac{\alpha + \beta}{2}$ (b) $\gamma = \alpha + \frac{\beta}{2}$

(c) $\gamma = \alpha$ (d) $\alpha < \gamma < \beta$

18. The equation $x - \frac{2}{x-1} = 1 - \frac{2}{x-1}$ has **[1984 - 2 Marks]**

(a) no root (b) one root
(c) two equal roots (d) infinitely many roots

19. If $(x^2 + px + 1)$ is a factor of $(ax^3 + bx + c)$, then **[1980]**

(a) $a^2 + c^2 = -ab$ (b) $a^2 - c^2 = -ab$
(c) $a^2 - c^2 = ab$ (d) none of these

20. Both the roots of the equation $(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b) = 0$ are always

(a) positive (b) real **[1980]**
(c) negative (d) none of these.

21. If ℓ, m, n are real, $\ell \ne m$, then the roots by the equation: $(\ell - m)x^2 - 5(\ell + m)x - 2(\ell - m) = 0$ are **[1979]**

(a) Real and equal (b) Complex
(c) Real and unequal (d) None of these

2 Integer Value Answer/ Non-Negative Integer

22. The product of all positive real values of x satisfying the equation $x^{\left(16(\log_5 x)^3 - 68 \log_5 x\right)} = 5^{-16}$ is ________ .

[Adv. 2022]

23. For $x \in \mathbb{R}$, then number of real roots of the equation $3x^2 - 4|x^2 - 1| + x - 1 = 0$ is ______. **[Adv. 2021]**

3 Numeric/ New Stem Based Questions

24. The smallest value of k, for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is **[2009]**

4 Fill in the Blanks

25. If the product of the roots of the equation $x^2 - 3kx + 2e^{2\ln k} - 1 = 0$ is 7, then the roots are real for $k = ...$ **[1984 - 2 Marks]**

26. If $2 + i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, where p and q are real, then $(p, q) = (..................., ...)$. **[1982 - 2 Marks]**

5 True / False

27. If $a < b < c < d$, then the roots of the equation $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are real and distinct. **[1984 - 1 Mark]**

28. The equation $2x^2 + 3x + 1 = 0$ has an irrational root. **[1983 - 1 Mark]**

6 MCQs with One or More than One Correct Answer

29. Let R^2 denote $R \times R$. Let $S = \{(a, b, c : a, b, c \in R$ and $ax^2 + 2bxy + cy^2 > 0$ for all $(x, y) \in R^2 - \{(0, 0)\}$.

Then which of the following statements is (are) TRUE? **[Adv. 2024]**

(a) $\left(2, \frac{7}{2}, 6\right) \in S$

(b) If $\left(3, b, \frac{1}{12}\right) \in S$, then $|2b| < 1$.

(c) For any given $(a, b, c) \in S$, the system of linear equations $ax + by = 1$ $by + cy = -1$ has a unique solution.

(d) For any given $(a, b, c) \in S$, the system of linear equations $(a+1)x + by = 0$ $bx + (c+1)y = 0$ has a unique solution.

30. If $3^x = 4^{x-1}$, then $x=$ **[Adv. 2013]**

(a) $\frac{2\log_3 2}{2\log_3 2-1}$ (b) $\frac{2}{2-\log_2 3}$

(c) $\frac{1}{1-\log_4 3}$ (d) $\frac{2\log_2 3}{2\log_2 3-1}$

8 Comprehension/Passage Based Questions

Let p, q be integers and let α, β be the roots of the equation, $x^2-x-1=0$, where $\alpha \neq \beta$. For $n=0, 1, 2, ...$, let $a_n = p\alpha^n + q\beta^n$.

FACT : If a and b are rational numbers and $a+b\sqrt{5}=0$, then $a=0=b$ **[Adv. 2017]**

31. $a_{12}=$

(a) $a_{11}-a_{10}$ (b) $a_{11}+a_{10}$

(c) $2a_{11}+a_{10}$ (d) $a_{11}+2a_{10}$

32. If $a_4=28$, then $p+2q=$

(a) 21 (b) 14 (c) 7 (d) 12

9 Assertion and Reason/Statement Type Questions

33. Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2+2px+q=0$ and $\alpha, \frac{1}{\beta}$ are the roots of the equation $ax^2+2bx+c=0$, where $\beta^2 \notin \{-1, 0, 1\}$

STATEMENT - 1 : $(p^2-q)(b^2-ac) \geq 0$

and

STATEMENT - 2 : $b \neq pa$ or $c \neq qa$ **[2008]**

(a) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1

(b) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is **NOT** a correct explanation for STATEMENT - 1

(c) STATEMENT - 1 is True, STATEMENT - 2 is False

(d) STATEMENT - 1 is False, STATEMENT - 2 is True

10 Subjective Problems

34. Let a and b be the roots of the equation $x^2-10cx-11d=0$ and those of $x^2-10ax-11b=0$ are c, d then the value of $a+b+c+d$, when $a \neq b \neq c \neq d$, is. **[2006 - 6M]**

35. If $x^2+(a-b)x+(1-a-b)=0$ where $a, b \in R$ then find the values of a for which equation has unequal real roots for all values of b. **[2003 - 4 Marks]**

36. If α, β are the roots of $ax^2+bx+c=0$, $(a \neq 0)$ and $\alpha+\delta, \beta+\delta$ are the roots of $Ax^2+Bx+C=0$, $(A \neq 0)$ for some constant δ, then prove that $\frac{b^2-4ac}{a^2} = \frac{B^2-4AC}{A^2}$. **[2000 - 4 Marks]**

37. Let a, b, c be real. If $ax^2+bx+c=0$ has two real roots α and β, where $\alpha < -1$ and $\beta > 1$, then show that $1+\frac{c}{a}+\left|\frac{b}{a}\right|<0$ **[1995 - 5 Marks]**

38. Solve $|x^2+4x+3|+2x+5=0$ **[1988 - 5 Marks]**

39. For $a \leq 0$, determine all real roots of the equation $x^2-2a|x-a|-3a^2=0$ **[1986 - 5 Marks]**

40. Solve for x; $(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$ **[1985 - 5 Marks]**

41. Solve the following equation for x : **[1978]**

$2\log_x a+\log_{ax} a+3\log_{a^2x} a=0$, $a>0$

42. Solve for x : $\sqrt{x+1}-\sqrt{x-1}=1$. **[1978]**

Topic-4: Condition for Common Roots, Maximum and Minimum value of Quadratic Equation, Quadratic Expression in two Variables, Solution of Quadratic Inequalities

1 MCQs with One Correct Answer

1. A value of b for which the equations
$x^2+bx-1=0$
$x^2+x+b=0$
have one root in common is **[2011]**
(a) $-\sqrt{2}$ (b) $-i\sqrt{3}$ (c) $i\sqrt{5}$ (d) $\sqrt{2}$

2. For all 'x', $x^2+2ax+10-3a>0$, then the interval in which 'a' lies is **[2004S]**
(a) $a<-5$ (b) $-5<a<2$
(c) $a>5$ (d) $2<a<5$

2 Integer Value Answer/ Non-Negative Integer

3. Let $f(x)=x^4+ax^3+bx^2+c$ be a polynomial with real coefficients such that $f(1)=-9$. Suppose that $i\sqrt{3}$ is a root of the equation $4x^3+3ax^2+2bx=0$ where $i=\sqrt{-1}$. If $\alpha_1, \alpha_2, \alpha_3$ and α_4 are all the roots of the equation $f(x)=0$, then $|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2$ is equal to ____. **[Adv. 2024]**

4 Fill in the Blanks

4. If the quadratic equations $x^2+ax+b=0$ and $x^2+bx+a=0$ $(a \neq b)$ have a common root, then the numerical value of $a+b$ is .... **[1986 - 2 Marks]**

5 True / False

5. If $P(x)=ax^2+bx+c$ and $Q(x)=-ax^2+dx+c$, where $ac \neq 0$, then $P(x)Q(x)=0$ has at least two real roots. **[1985 - 1 Mark]**

6 MCQs with One or More than One Correct Answer

6. Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2-x+\alpha=0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1-x_2|<1$. Which of the following intervals is(are) a subset(s) of S? **[JEE Adv. 2015]**
(a) $\left(-\frac{1}{2},-\frac{1}{\sqrt{5}}\right)$ (b) $\left(-\frac{1}{\sqrt{5}},0\right)$
(c) $\left(0,\frac{1}{\sqrt{5}}\right)$ (d) $\left(\frac{1}{\sqrt{5}},\frac{1}{2}\right)$

7. If a, b, c, d and p are distinct real numbers such that $(a^2+b^2+c^2)p^2-2(ab+bc+cd)p+(b^2+c^2+d^2)\leq 0$ then a, b, c, d **[1987 - 2 Marks]**
(a) are in A. P. (b) are in G. P.
(c) are in H. P. (d) satisfy $ab=cd$
(e) satisfy none of these

8. For real x, the function $\frac{(x-a)(x-b)}{x-c}$ will assume all real values provided **[1984 - 3 Marks]**
(a) $a>b>c$ (b) $a<b<c$
(c) $a>c>b$ (d) $a<c<b$

10 Subjective Problems

9. Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equation $ax^2+bx+c=0$. Express the roots of $a^3x^2+abcx+c^3=0$ in terms of α, β. **[2001 - 4 Marks]**

10. Find all real values of x which satisfy $x^2-3x+2>0$ and $x^2-3x-4\leq 0$ **[1983 - 2 Marks]**

11. If α, β are the roots of $x^2+px+q=0$ and γ, δ are the roots of $x^2+rx+s=0$, then evaluate $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$ in terms of p, q, r and s.
Deduce the condition that the equations have a common root. **[1979]**

Answer Key

Topic-1 : Integral Powers of Iota, Algebraic Operations of Complex Numbers, Conjugate, Modulus and Argument or Amplitude of a Complex Number

1. (d) 2. (b) 3. (a) 4. (a) 5. (d) 6. (c) 7. (d) 8. (d) 9. (281) 10. (4)
11. (5) 12. (0.50) 13. (4) 14. $\left(2p\pi, n\pi+\frac{\pi}{4}\right)$ 15. (True) 16. (a, c, d) 17. (a) 18. (b, c) 19. (a, c, d)
20. (a, b, d) 21. (a, b) 22. (a,c,d) 23. (d) 24. (b) 25. (d) 26. (c) 27. (a, d) 28. (a, b, c)
29. (b) 30. (c) 31. (b) 32. (c) 33. (b) 34. (c) 35. (d)

Topic-2 : Rotational Theorem, Square Root of a Complex Number, Cube Roots of Unity, Geometry of Complex Numbers, De-moiver's Theorem, Powers of Complex Numbers

1. (c) 2. (d) 3. (c) 4. (d) 5. (a) 6. (d) 7. (d) 8. (d) 9. (d) 10. (b)
11. (a) 12. (b) 13. (a) 14. (b) 15. (c) 16. (d) 17. (c) 18. (b) 19. (b) 20. (b)
21. (b) 22. (d) 23. (b) 24. (a) 25. (b) 26. (8) 27. (3) 28. $\frac{1}{4}(n-1)n(n^2+3n+4)$
29. $-2, 1-i\sqrt{3}$ 30. $3-i/2$ or $1-\frac{3}{2}i$ 31. $2-\sqrt{3}, 2-\sqrt{3}$ 32. $(a^2+b^2)(|z_1|^2+|z_2|^2)$ 33. (True) 34. (False)
35. (True) 36. (a, c, d) 37. (c, d) 38. (d) 39. A → (q, r); B → (p); C → (p, s, t); D → (q, r, s, t)
40. A → (q); B → (p)

Topic-3 : Solutions of Quadratic Equations, Sum and Product of Roots, Nature of Roots, Relation Between Roots and Co-efficients, Formation of an Equation with Given Roots

1. (d) 2. (c) 3. (d) 4. (c) 5. (c) 6. (c) 7. (b) 8. (d) 9. (a) 10. (a)
11. (c) 12. (d) 13. (b) 14. (a) 15. (c) 16. (c) 17. (d) 18. (a) 19. (c) 20. (b)
21. (c) 22. (1) 23. (4) 24. (2) 25. (2) 26. (–4, 7) 27. (True) 28. (False) 29. (a,b,c) 30. (a,b,c)
31. (b) 32. (d) 33. (b)

Topic-4 : Condition for Common Roots, Maximum and Minimum value of Quadratic Equation, Quadratic Expression in two Variables, Solution of Quadratic Inequalities

1. (b) 2. (b) 3. (20) 4. (1) 5. (True) 6. (a, d) 7. (b) 8. (c, d)

Hints & Solutions

Topic-1: Factorials and Permutations

1. (d) $\because \dfrac{w-\bar{w}z}{1-z}$ is purely real

$\therefore \left(\dfrac{w-\bar{w}z}{1-z}\right)=\overline{\left(\dfrac{w-\bar{w}z}{1-z}\right)} \Rightarrow \dfrac{\bar{w}-w\bar{z}}{1-\bar{z}}=\dfrac{w-\bar{w}z}{1-z}$

$\Rightarrow \bar{w}-\bar{w}z-w\bar{z}+wz\bar{z}=w-w\bar{z}-\bar{w}z+\bar{w}z\bar{z}$

$\Rightarrow w-\bar{w}=(w-\bar{w})|z|^2$

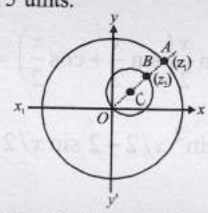
$\Rightarrow |z|^2=1$ ($\because w=\alpha+i\beta$ and $\beta\neq 0$)

$\Rightarrow |z|=1$ and also given that $z\neq 1$

$\therefore$ The required set is $\{z:|z|=1, z\neq 1\}$ $=3\omega(\omega-1)$

2. (b) $|z_1|=12 \Rightarrow z_1$ lies on a circle with centre (0, 0) and radius 12 units.

And $|z_2-3-4i|=5 \Rightarrow z_2$ lies on a circle with centre (3, 4) and radius 5 units.



From figure, it is clear that $|z_1-z_2|$ i.e., distance between z_1 and z_2 will be min when they lie at A and B respectively i.e., O, C, B, A are collinear as shown.

Then $z_1-z_2=AB=OA-OB=12-2(5)=2$. As above is the minimum value, we must have $|z_1-z_2|\geq 2$.

3. (a) Given : $|z_1|=|z_2|=|z_3|=1$

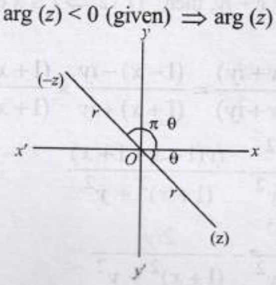
Now, $|z_1|=1 \Rightarrow |z_1|^2=1 \Rightarrow z_1\bar{z}_1=1$

Similarly $z_2\bar{z}_2=1,\ z_3\bar{z}_3=1$

Now, $\left|\dfrac{1}{z_1}+\dfrac{1}{z_2}+\dfrac{1}{z_3}\right|=1 \Rightarrow |\bar{z}_1+\bar{z}_2+\bar{z}_3|=1$

$\Rightarrow |\overline{z_1+z_2+z_3}|=1 \Rightarrow |z_1+z_2+z_3|=1$

4. (a) Given : arg $(z)<0$ (given) $\Rightarrow$ arg $(z)=-\theta$



Now, $z=r\cos(-\theta)+i\sin(-\theta)=r[\cos(\theta)-i\sin(\theta)]$

Again $-z=-r[\cos(\theta)-i\sin(\theta)]$

$=r[\cos(\pi-\theta)+i\sin(\pi-\theta)]$

$\therefore$ arg $(-z)=\pi-\theta$;

Thus arg $(-z)-\arg(z)=\pi-\theta-(-\theta)=\pi-\theta+\theta=\pi$

5. (d) $(1+i)^{n_1}+(1+i^3)^{n_1}+(1+i^5)^{n_2}+(1+i^7)^{n_2}$

$=(1+i)^{n_1}+(1-i)^{n_1}+(1+i)^{n_2}+(1-i)^{n_2}$

Using $1+i=\sqrt{2}(\cos\pi/4+i\sin\pi/4)$

and $1-i=\sqrt{2}(\cos\pi/4-i\sin\pi/4)$

We get the given expression as

$$=(\sqrt{2})^{n_1}\left[\cos\frac{n_1\pi}{4}+i\sin\frac{n_1\pi}{4}\right]$$

$$+(\sqrt{2})^{n_2}\left[\cos\frac{n_2\pi}{4}+i\sin\frac{n_2\pi}{4}\right]$$

$$+(\sqrt{2})^{n_2}\left[\cos\frac{n_2\pi}{4}-i\sin\frac{n_2\pi}{4}\right]$$

$$=(\sqrt{2})^{n_1}\left[2\cos\frac{n_1\pi}{4}\right]+(\sqrt{2})^{n_2}\left[2\cos\frac{n_2\pi}{4}\right]$$

= real number irrespective the values of n_1 and n_2

$\therefore$ (d) is the most appropriate answer.

6. (c) Given that $|z+i\omega|=|z-i\bar{\omega}|$

$\Rightarrow |z-(-i\omega)|=|z-(-i\bar{\omega})|$

$\Rightarrow z$ lies on perpendicular bisector of the line segment joining $(-i\omega)$ and $(-i\bar{\omega})$, which is real axis, $(-i\omega)$ and $(-i\bar{\omega})$ being mirror images of each other.

$\therefore$ Im$(z)=0$.

If $z=x$, then $|z|\leq 1 \Rightarrow x^2\leq 1 \Rightarrow -1\leq x\leq 1$

$\therefore$ (c) is the correct option.

7. (d) $\because |z|=|\omega|$ and arg $z=\pi-\arg\omega$

Let $\omega=re^{i\theta}$ then $z=re^{i(\pi-\theta)}$

$\Rightarrow z=re^{i\pi}.e^{-i\theta}$

$=(re^{-i\theta})(\cos\pi+i\sin\pi)=\bar{\omega}(-1)=-\bar{\omega}$

8. (d) $\dfrac{1+i}{1-i}=\dfrac{(1+i)^2}{(1-i)(1+i)}=\dfrac{1-1+2i}{2}=i$

Now $i^n=1 \Rightarrow$ the smallest positive integral value of n should be 4.

9. (281) is a positive integer

$$=\frac{281(49+18\sin\theta.\cos\theta+i(21\cos\theta+42\sin\theta))}{49+9\cos^2\theta}$$

for postive integer Im(z) = 0

$21\cos\theta+42\sin\theta=0$

$\Rightarrow\ \tan\theta = -\frac{1}{2}, \sin 2\theta = \frac{-4}{5}, \cos^2\theta = \frac{4}{5}.$

Now $\text{Re}(2) = \frac{281(49 - 9\sin 2\theta)}{49 + 9\cos^2\theta}$

$= \frac{281\left(49 - 9\times -\frac{4}{5}\right)}{49 + 9\times\frac{4}{5}} = 281.$

10. (4) Given : $\alpha_k = \cos\frac{k\pi}{7} + i\sin\frac{k\pi}{7} = e^{\frac{i\pi k}{7}}$

$\alpha_{k+1} - \alpha_k = e^{\frac{i\pi(k+1)}{7}} - e^{\frac{i\pi k}{7}} = e^{\frac{i\pi k}{7}}(e^{i\pi/7} - 1)$

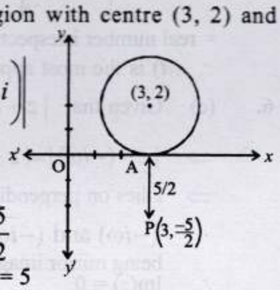
$|\alpha_{k+1} - \alpha_k| = |e^{i\pi/7} - 1|$

$\Rightarrow\ \sum_{k=1}^{12}|\alpha_{k+1} - \alpha_k| = 12|e^{i\pi/7} - 1|$

Similarly, $\sum_{k=1}^{3}|\alpha_{4k-1} - \alpha_{4k-2}| = 3|e^{i\pi/7} - 1|$

$\therefore\ \frac{\sum_{k=1}^{12}|\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1} - \alpha_{4k-2}|} = 4$

11. (5) Given : $|z - 3 - 2i| \le 2,$ which represents a circular region with centre (3, 2) and radius 2.



Now, $|2z - 6 + 5i| = 2\left|z - \left(3 - \frac{5}{2}i\right)\right|$

= 2 × distance of z from P

(where z lies in or on the circle)

Also min distance of z from $P = \frac{5}{2}$

$\therefore$ Minimum value of $|2z - 6 + 5i| = 5$

12. (0.50) Let $X = \frac{4z^2 + 3z + 2}{4z^2 - 3z + 2}$

It can be written as

$= 1 + \frac{6z}{4z^2 - 3z + 2}$

Now $X = 1 + \frac{6}{2\left(2z + \frac{1}{z}\right) - 3}$

$\because X \in R$, then $2z + \frac{1}{z} \in R$

$\Rightarrow\ 2z + \frac{1}{z} = 2\bar{z} + \frac{1}{\bar{z}} \Rightarrow 2(z - \bar{z}) - \frac{z - \bar{z}}{|z|^2} = 0$

$\because\ (z - \bar{z})\left(2 - \frac{1}{|z|^2}\right) = 0$

$\because\ z \ne \bar{z}$ (given). So, $|z|^2 = \frac{1}{2}$

13. (4) Given, $\bar{z} - z^2 = i\left(\bar{z} + z^2\right)$

It can be written as $\bar{z}(1 - i) = z^2(1 + i)$

So $|\bar{z}||1 - i| = |z|^2|1 + i|$

$|z| = |z|^2 \Rightarrow |z| = 0$ or $|z| = 1$

Let arg $(z) = \alpha$. So from (i), we get

$2n\pi - \alpha - \frac{\pi}{4} = 2\alpha + \frac{\pi}{4}$

$\Rightarrow\ \alpha = \frac{1}{3}\left(\frac{4n - 1}{2}\right)\pi = \frac{(4n - 1)\pi}{6}$

So we will get 3 distinct values of α. Hence there will be total 4 possible values of complex number z.

14. Let $z = \frac{\sin x/2 + \cos x/2 + i\tan x}{1 + 2i\sin x/2}$

$= \frac{(\sin x/2 + \cos x/2 + i\tan x)(1 - 2i\sin x/2)}{(1 + 2i\sin x/2)(1 - 2i\sin x/2)}$

$= \frac{[\sin x/2 + \cos x/2 + 2\sin x/2\tan x) + i(\tan x - 2\sin^2 x/2 - 2\sin x/2\cos x/2)]}{(1 + 4\sin^2 x/2)}$

But it is given that z is real.

$\therefore\ I_m(z) = 0$

$\Rightarrow\ \tan x - 2\sin\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) = 0$

$\Rightarrow\ \frac{\sin x}{\cos x} - 2\sin^2 x/2 - 2\sin x/2\cos x/2 = 0$

$\Rightarrow\ \frac{\sin x}{\cos x} - (1 - \cos x) - \sin x = 0$

$\Rightarrow\ \sin x\left[\frac{1}{\cos x} - 1\right] - [1 - \cos x] = 0$

$\Rightarrow\ \left(\frac{1 - \cos x}{\cos x}\right)\sin x - [1 - \cos x] = 0$

$\Rightarrow\ (1 - \cos x)\left(\frac{\sin x}{\cos x} - 1\right) = 0$

$\Rightarrow\ \cos x = 1\ \Rightarrow\ x = 2n\pi$

and $\tan x = 1\ \Rightarrow\ x = n\pi + \pi/4$

$\therefore\ x = 2n\pi, n\pi + \pi/4$

15. (True) Let $z = x + iy$, then $1 \cap z \Rightarrow 1 \le x \,\&\, 0 \le y$ (by def.)

Consider,

$\frac{1 - z}{1 + z} = \frac{1 - (x + iy)}{1 + (x + iy)} = \frac{(1 - x) - iy}{(1 + x) + iy} \times \frac{(1 + x) - iy}{(1 + x) - iy}$

$= \frac{1 - x^2 - y^2}{(1 + x)^2 + y^2} - \frac{iy(1 - x + 1 + x)}{(1 + x)^2 + y^2}$

$= \frac{1 - x^2 - y^2}{(1 + x)^2 + y^2} - \frac{2iy}{(1 + x)^2 + y^2}$

$\frac{1-z}{1+z} \cap 0 \Rightarrow \frac{1-x^2-y^2}{(1+x)^2+y^2} \le 0$

and $\frac{-2y}{(1+x)^2+y^2} \le 0$

$\Rightarrow 1-x^2-y^2 \le 0$ and $-2y \le 0$

$\Rightarrow x^2+y^2 \ge 1$ and $y \ge 0$, which is true as

$x \ge 1$ and $y \ge 0$

Hence, the given statement is true $\forall z \in C$.

16. **(a, c, d)** (a) S = $\{a+b\sqrt{2} : a, b \in Z\}$

For $b = 0$; $Z \subset S$

$T_1 = (-1+\sqrt{2})^n = m+\sqrt{2}n, m, n \in Z$

$T_2 = (1+\sqrt{2})^n = m_1+\sqrt{2}n_1, m_1, n, \in Z$

For $n \in N$ elements of T_1 and T_2 are of the form $a+b\sqrt{2}$

Hence $Z \cup T_1 \cup T_2 \subset S$

(b) Now, $-1+\sqrt{2} < 1$ and its higher powers decreases

$\Rightarrow (-1+\sqrt{2})^n < 1$ and can be made in $\left(0, \frac{1}{2024}\right)$ for some higher n.

(c) $1+\sqrt{2} > 1$ and its higher power increases

$\Rightarrow (1+\sqrt{2})^n$ can be made in $(2024, \infty)$ for some higher n.

(d) $\cos\pi(a+b\sqrt{2})+i\sin\pi(a+b\sqrt{2}) \in Z$ if

$a+b\sqrt{2}$ is an integer $\Rightarrow b = 0$

17. **(a)** Let $z = r.e^{i\theta} \Rightarrow \bar{z} = re^{-i\theta}$

$\therefore (\bar{z})^2+\frac{1}{z^2} = r^2e^{-2i\theta}+\frac{1}{r^2e^{2i\theta}} = \left(r^2+\frac{1}{r^2}\right)e^{-2i\theta} = a+ib$ (say),

where a, b $\in$ Z

So, $\left(r^2+\frac{1}{r^2}\right)^2 = a^2+b^2 \Rightarrow r^8-(a^2+b^2-2)r^4+1 = 0$

$\Rightarrow r^4 = \frac{(a^2+b^2-2) \pm \sqrt{(a^2+b^2-2)^2-4}}{2}$

for option (a): $|z|^4 = \frac{43+3\sqrt{205}}{2}$

$\Rightarrow a^2+b^2 = 45$ i.e. $(a, b) = (\pm 6, \pm 3)$ or $(\pm 3, \pm 6)$

For option (b): $|z|^4 = \frac{7+\sqrt{33}}{4} \Rightarrow a^2+b^2 = \frac{11}{2}$

For option (c): $a^2+b^2 = \frac{13}{2}$

For option (d): $a^2+b^2 = \frac{13}{3}$

18. **(b, c)** $|z^2+z+1| = 1$

$\Rightarrow \left|\left(z+\frac{1}{2}\right)^2+\frac{3}{4}\right| \ge 1 \Rightarrow \left|\left(z+\frac{1}{2}\right)\right|^2 \ge \frac{1}{4} \Rightarrow \left|z+\frac{1}{2}\right| \ge \frac{1}{2}$

also $|(z^2+z)+1| = 1$

$\Rightarrow |z^2+z|-1 \le 1 \Rightarrow |z^2+z| \le 2$

$\Rightarrow ||z^2|-|z|| \le |z^2+z| \le 2 \Rightarrow |r^2-r| \le 2 \Rightarrow r = |z| \le 2;\ \forall z \in S$

Hence, set 'S' is infinite

19. **(a, c, d)**

We have,

$sz+t\bar{z}+r = 0$ ..... (i)

On taking conjugate

$\bar{s}\bar{z}+\bar{t}z+\bar{r} = 0$ ..... (ii)

On solving Eqs. (i) and (ii), we get

$z = \frac{\bar{r}t-r\bar{s}}{|s|^2-|t|^2}$

(a) For unique solutions of z

$|s|^2-|t|^2 \ne 0 \Rightarrow |s| \ne |t|$

It is true.

(b) If $|s| = |t|$, then $\bar{r}t-r\bar{s}$ may or may not be zero.

So, z may have no solution.

$\therefore$ L may be an empty set.

It is false.

(c) If elements of set L represents line, then this line and given circle intersect at maximum two point.

Hence, it is true.

(d) In the case locus of z is a line, so L has infinite elements.

Hence, it is true.

20. **(a, b, d)**

(a) $\arg(-1-i) = \frac{-3\pi}{4}$

$\therefore$ (a) is false

(b) $f(t) = \arg(-1+it) = \begin{cases} \pi-\tan^{-1}(t), t \ge 0 \\ -\pi+\tan^{-1}(t), t < 0 \end{cases}$

$\lim_{t \to 0^-} f(t) = -\pi$ and $\lim_{t \to 0^+} f(t) = \pi$

LHL $\ne$ RHL $\Rightarrow$ f is discontinuous at $t = 0$

$\therefore$ (b) is false.

(c) $\arg\left(\frac{z_1}{z_2}\right)-\arg z_1+\arg z_2$

$= 2n\pi+\arg z_1-\arg z_2-\arg z_1+\arg z_2$

$= 2n\pi$, multiple of 2π

$\therefore$ (c) is true.

(d) $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$

$\Rightarrow \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = k, \quad k \in R$

$\Rightarrow \left(\frac{z-z_1}{z-z_3}\right)=k\left(\frac{z_2-z_1}{z_2-z_3}\right)$

$\Rightarrow$ z, z_1, z_2, z_3 are concyclic. i.e. z lies on a circle.

$\therefore$ (d) is false.

21. **(a, b)** $a-b=1, y\neq 0$

$\text{Im}\left(\frac{az+b}{z+1}\right)=y$

$\Rightarrow \text{Im}\left[\frac{a(x+iy)+b}{(x+1)+iy}\times\frac{(x+1)-iy}{(x+1)-iy}\right]=y$

$\Rightarrow \frac{-(ax+b)y+ay(x+1)}{(x+1)^2+y^2}=y$

$\Rightarrow \frac{-axy-by+axy+ay}{(x+1)^2+y^2}=y$

$\Rightarrow a-b=(x+1)^2+y^2$

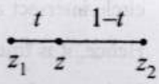
$\Rightarrow 1=(x+1)^2+y^2, \therefore x=-1\pm\sqrt{1-y^2}$

22. **(a,c,d)** Given : $z=(1-t)z_1+tz_2$, where $0<t<1$

$\Rightarrow z=\frac{(1-t)z_1+tz_2}{(1-t)+t}$

$\Rightarrow$ z divides the join of z_1 and z_2 internally in the ratio $t:(1-t)$.

$\therefore$ z_1, z and z_2 are collinear



$\Rightarrow |z-z_1|+|z-z_2|=|z_1-z_2|$

Also $z=(1-t)z_1+t\,z_2$

$\Rightarrow \frac{z-z_1}{z_2-z_1}=t$, which is purely real number

$\therefore \arg\left(\frac{z-z_1}{z_2-z_1}\right)=0\Rightarrow \arg(z-z_1)=\arg(z_2-z_1)$

Also $\frac{z-z_1}{z_2-z_1}=t\Rightarrow\frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}=t$

$\Rightarrow \frac{z-z_1}{z_2-z_1}=\frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$

$\Rightarrow (z-z_1)(\bar{z}_2-\bar{z}_1)=(\bar{z}-\bar{z}_1)(z_2-z_1)$

$\Rightarrow \begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1\end{vmatrix}=0$

23. **(d)** Taking $-3i$ common from C_2, we get

$-3i\begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i\end{vmatrix}=0 \quad (\because C_2\equiv C_3)$

$\Rightarrow x=0,\ y=0$

24. **(b)** $\sum_{i=1}^{13}(i^n+i^{n+1})=\sum_{i=1}^{13}i^n(1+i)=(1+i)\sum_{i=1}^{13}i^n$,

Which forms a G.P.

Sum of G.P. $=i\,(1+i)\,\frac{(1-i^{13})}{1-i}=i-1$ as $i^{13}=i$

25. **(d)** Let $z=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$

By DeMoivre's theorem,

$z^k=\cos\frac{2\pi k}{7}+i\sin\frac{2\pi k}{7}$

Now, $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}-i\cos\frac{2\pi k}{7}\right)$

$=\sum_{k=1}^{6}(-i)\left(\cos\frac{2\pi k}{7}+i\sin\frac{2\pi k}{7}\right)$

$=(-i)\sum_{k=1}^{6}z^k=-i\,z\frac{(1-z^6)}{1-z}=-i\left(\frac{z-z^7}{1-z}\right)$

$=(-i)\left(\frac{z-1}{1-z}\right)=[\because z^7=\cos 2\pi+i\sin 2\pi=1]$

$=i\left(\frac{1-z}{1-z}\right)=i$

26. **(c)** Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$

and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$

where $r_1=|z_1|, r_2=|z_2|, \theta_1=\arg(z_1), \theta_2=\arg(z_2)$

$\therefore z_1+z_2=r_1(\cos\theta_1+i\sin\theta_1)+r_2(\cos\theta_2+i\sin\theta_2)$

$=(r_1\cos\theta_1+r_2\cos\theta_2)+i(r_1\sin\theta_1+r_2\sin\theta_2)$

So, $|z_1+z_2|=r_1^2\cos^2\theta_1+r_2^2\cos^2\theta_2+2r_1r_2\cos\theta_1\cos\theta_2$

$+r_1^2\sin^2\theta_1+r_2^2\sin^2\theta_2+2r_1r_2\sin\theta_1\sin\theta_2$

$=r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)$

and $|z_1|+|z_2|=r_1+r_2$

Given $|z_1+z_2|=|z_1|+|z_2|$

$\Rightarrow |z_1+z_2|^2=|z_1|^2+|z_2|^2+2|z_1||z_2|$

$\Rightarrow r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)=r_1^2+r_2^2+2r_1r_2$

$\Rightarrow \cos(\theta_1-\theta_2)=1\Rightarrow\theta_1-\theta_2=0$

$\therefore \arg(z_1)=\arg(z_2)$

27. **(a, d)** Let $z_1=a+ib, a>0$ and $b\in R$; $z_2=c+id$, $d<0, c\in R$, then

$|z_1|=|z_2|\Rightarrow a^2+b^2=c^2+d^2$

$\Rightarrow a^2-c^2=d^2-b^2$ ....(i)

Now, $\frac{z_1+z_2}{z_1-z_2}=\frac{(a+c)+i(b+d)}{(a-c)+i(b-d)}$

$=\frac{[(a^2-c^2)+(b^2-d^2)]+i[(a-c)(b+d)-(a+c)(b-d)]}{(a-c)^2+(b-d)^2}$

$= \dfrac{i[(a-c)(b+d)-(a+c)(b-d)]}{(a-c)^2+(b-d)^2}$ [Using (i)]

Which is purely imaginary number or zero in case $a+c=b+d=0$.

28. **(a, b, c)** $z_1 = a+ib$ and $z_2 = c+id$.

Acc. to the ques, $|zi|^2 = |z_2|^2 = 1$

$\Rightarrow a^2+b^2=1$ and $c^2+d^2=1$. ....(i)

Also Re $(z_1 \overline{z_2}) = 0 \Rightarrow ac+bd=0$

$\Rightarrow \dfrac{a}{b} = \dfrac{-d}{c} = \alpha\ (say)$ ....(ii)

From (i) and (ii), we get

$b^2\alpha^2 + b^2 = c^2\alpha^2 + c^2 \Rightarrow b^2 = c^2;$

Similarly, $a^2 = d^2$

$\therefore\ |\omega_1| = \sqrt{a^2+c^2} = \sqrt{c^2+d^2} = 1$

and $|\omega_2| = \sqrt{b^2+d^2} = \sqrt{c^2+d^2} = 1$

Also, Re $(\omega_1\bar{\omega}_2) = ab+cd = (b\alpha)b + c(-c\alpha)$

$= \alpha(b^2-c^2) = 0$

29. **(b)** Given, $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$ ...(i)

$|\bar{z}|^3 + 2\bar{z}^2 + 4z - 8 = 0$ [Conjugate both sides]

$2(z^2-\bar{z}^2) + 4(\bar{z}-z) = 0$

$\Rightarrow 2(z-\bar{z})[z+\bar{z}-2] = 0$

$\because z=\bar{z}$ (Not possible) or $z+\bar{z}=2$

$\therefore z = 1+bi\ (b\neq 0) \Rightarrow \bar{z} = 1-bi$

$(1+b^2)^{3/2} + 2(1-b^2+2bi) + 4(1-bi) - 8 = 0$ [from (i)]

$(1+b^2)^{3/2} - 2(1+b^2) = 0$

$\Rightarrow (1+b^2)\left(\sqrt{1+b^2}-2\right) = 0$

$\because 1+b^2 \neq 0 \Rightarrow \sqrt{1+b^2}-2=0 \Rightarrow b^2 = 3$

(P) $|z|^2 = 1+b^2 = 1+3 = 4$

(Q) $|z-\bar{z}|^2 = |1+ib-1+ib|^2 = 4b^2 = 12$

(R) $|z|^2 + |z+\bar{z}|^2 = 4 + |1+ib+1-ib|^2 = 4+4 = 8$

(S) $|z+1|^2 = |1+1+ib|^2 = 4+b^2 = 4+3 = 7.$

30. **(c)** (P) $\to$ (1): $z_k = \cos\dfrac{2k\pi}{10} + i\sin\dfrac{2k\pi}{10}$, $k = 1$ to 9

$\therefore\ z_k = e^{i\frac{2k\pi}{10}}$

Now $z_k z_j = 1 \Rightarrow z_j = \dfrac{1}{z_k} = e^{-i\frac{2k\pi}{10}} = \overline{z_k}$

We know if z_k is 10th root of unity so will be $\overline{z}_k$.

$\therefore$ For every z_k, there exist $z_i = \overline{z}_k$

Such that $z_k.z_j = z_k.\overline{z}k = 1$

Hence the statement is true.

(Q) $\to$ (2) $z_1 = z_k \Rightarrow z = \dfrac{z_k}{z_1}$ for $z_1 \neq 0$

$\therefore$ We can always find a solution of $z_1.z = z_k$

Hence the statement is false.

(R) $\to$ (3) : We know $z^{10} - 1 = (z-1)(z-z_1)....(z-z_9)$

$\Rightarrow (z-z_1)(z-z_2).....(z-z_9) = \dfrac{z^{10}-1}{z-1}$

$= 1 + z + z^2 + ... z^9$

For z = 1, we get $(1-z_1)(1-z_2).....(1-z_9) = 10$

$\therefore\ \dfrac{|1-z_1||1-z_2|.....|1-z_9|}{10} = 1$

(S) $\to$ (4) : 1, $Z_1, Z_2, ..., Z_9$ are 10th roots of unity.

$\therefore Z^{10} - 1 = 0$

From equation $1 + Z_1 + Z_2 + ... + Z_9 = 0$,

Re (1) + Re (Z_1) + Re (Z_2) + ... + Re(Z_9) = 0

$\Rightarrow$ Re (Z_1) + Re (Z_2) + ... Re(Z_9) = – 1

$\Rightarrow \sum_{K=1}^{9} \cos\dfrac{2k\pi}{10} = -1 \Rightarrow 1 - \sum_{K=1}^{9}\cos\dfrac{2k\pi}{10} = 2$

Hence (c) is the correct option.

For (Qs. 31-32)

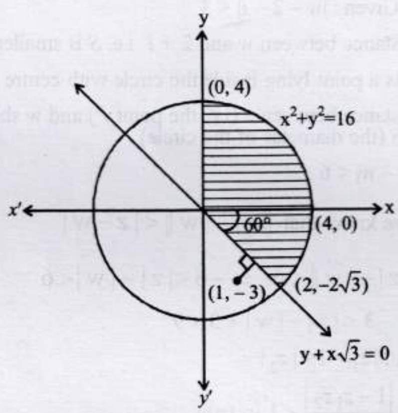
$S_1 : x^2+y^2 < 16$

$S_2 : \text{Im}\left[\dfrac{(x-1)+i(y+\sqrt{3})}{1-i\sqrt{3}}\right] > 0$

$\Rightarrow \sqrt{3}(x-1) + (y+\sqrt{3}) > 0 \Rightarrow y+\sqrt{3}x > 0$

$S_3 : x > 0$

Then S : $S_1 \cap S_2 \cap S_3$ is as shown in the figure given below.



31. **(b)** Area of shaded region

$= \dfrac{\pi}{4}\times 4^2 + \dfrac{\pi\times 4^2\times 60^\circ}{360^\circ} = 4\pi + \dfrac{8\pi}{3} = \dfrac{20\pi}{3}$

32. **(c)** $\min\limits_{z\in s}|1-3i-z|$ = min distance between z and (1, –3)

Clearly (from figure) minimum distance between z∈S and (1, –3)

from line $y+x\sqrt{3}=0$ i.e. $\left|\dfrac{\sqrt{3}-3}{\sqrt{3+1}}\right| = \dfrac{3-\sqrt{3}}{2}$

For (Qs. 33 - 35)

Given : $A = \{z : \text{Im}(z) \geq 1\} = \{(x, y) : y \geq 1\}$

Clearly A is the set of all points lying on or above the line $y = 1$ in cartesian plane.

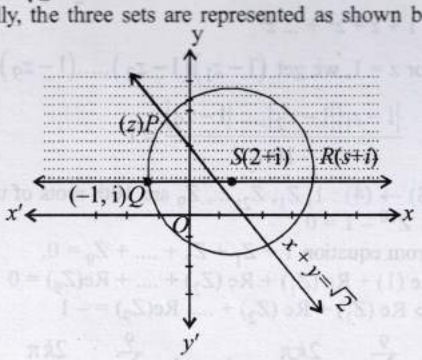
$B = \{z : |z - 2 - i| = 3\} = \{(x, y) : (x-2)^2 + (y-1)^2 = 9\}$

$\Rightarrow B$ is the set of all points lying on the boundary of the circle with centre (2, 1) and radius 3.

$C = \{z : \text{Re}[(1-i)z] = \sqrt{2}\} = \{(x, y) : x + y = \sqrt{2}\}$

$\Rightarrow$ C is the set of all points lying on the straight line represented by $x + y = \sqrt{2}$.

Graphically, the three sets are represented as shown below :



33. (b) From graph $A \cap B \cap C$ consists of only one point P [the common point of the region $y \geq 1$, $(x-2)^2 + (y-1)^2 = 9$ and $x + y = \sqrt{2}$] $\therefore n(A \cap B \cap C) = 1$

34. (c) Since, z is a point of $A \cap B \cap C \Rightarrow z$ represents the point P

$\therefore |z + 1 - i|^2 + |z - 5 - i|^2$

$\Rightarrow |z - (-1+i)|^2 + |z - (5+i)|^2$

$\Rightarrow PQ^2 + PR^2 = QR^2 = 6^2 = 36$, which lies between 35 and 39

$\therefore$ (c) is correct option.

35. (d) Given : $|w - 2 - i| < 3$

$\Rightarrow$ Distance between w and $2 + i$ i.e. S is smaller than 3.

$\Rightarrow w$ is a point lying inside the circle with centre S and radius 3.

$\Rightarrow$ Distance between z (i.e. the point P) and w should be smaller than 6 (the diameter of the circle)

i.e. $|z - w| < 6$

But we know that $||z| - |w|| < |z - w|$

$\Rightarrow ||z| - |w|| < 6 \Rightarrow -6 < |z| - |w| < 6$

$-3 < |z| - |w| + 3 < 9$

36. Given : $|z_1| < 1 < |z_2|$

Then $\left|\dfrac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$ is true

if $|1 - z_1\bar{z}_2| < |z_1 - z_2|$ is true

or if $|1 - z_1\bar{z}_2|^2 < |z_1 - z_2|^2$ is true

or if $(1 - z_1\bar{z}_2)\overline{(1 - z_1\bar{z}_2)} < (z_1 - z_2)\overline{(z_1 - z_2)}$ is true

or if $(1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$ is true

or if $1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_1\bar{z}_1 z_2\bar{z}_2 < z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_2\bar{z}_2$ is true

or, if $1 + |z_1|^2|z_2|^2 < |z_1|^2 + |z_2|^2$ is true

or, if $(1 - |z_1|^2)(1 - |z_2|^2) < 0$ is true.

which is obviously true

as $|z_1| < 1 < |z_2| \Rightarrow |z_1|^2 < 1 < |z_2|^2$

$\Rightarrow |1 - |z_1|^2 > 0$ and $(1 - |z_2|^2) < 0$

37. Given : $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$

Also $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \dfrac{\pi}{4}$

$\Rightarrow \arg(z - z_1) - \arg(z - z_2) = \dfrac{\pi}{4}$

$\Rightarrow \arg((x+iy) - (10+6i)) - \arg((x+iy) - (4+6i)) = \dfrac{\pi}{4}$

$\Rightarrow \arg[(x-10) + i(y-6)] - \arg[(x-4) + i(y-6)] = \dfrac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\dfrac{y-6}{x-10}\right) - \tan^{-1}\left(\dfrac{y-6}{x-4}\right) = \dfrac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\dfrac{\dfrac{y-6}{x-10} - \dfrac{y-6}{x-4}}{1 + \dfrac{(y-6)^2}{(x-4)(x-10)}}\right) = \dfrac{\pi}{4}$

$\Rightarrow \dfrac{(x-4)(y-6) - (x-10)(y-6)}{(x-4)(x-10) + (y-6)^2} = \tan\dfrac{\pi}{4}$

$\Rightarrow (x - 4 - x + 10)(y - 6) = (x-4)(x-10) + (y-6)^2$

$\Rightarrow 6y - 36 = x^2 + y^2 - 14x - 12y + 40 + 36$

$\Rightarrow x^2 + y^2 - 14x - 18y + 112 = 0$

$\Rightarrow (x^2 - 14x + 49) + (y^2 - 18y + 81) = 18$

$\Rightarrow (x-7)^2 + (y-9)^2 = (3\sqrt{2})^2$

$\Rightarrow |(x+iy) - (7+9i)| = 3\sqrt{2}$

$\Rightarrow |z - (7+9i)| = 3\sqrt{2}$

38. Let $A = z = x + iy$, $B = iz = -y + ix$, $C = z + iz = (x - y) + i(x + y)$

Now, area of $\Delta ABC = \dfrac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ x-y & x+y & 1 \end{vmatrix}$

On applying, $R_2 - R_1$, $R_3 - R_1$, we get

$\Delta = \dfrac{1}{2}\begin{vmatrix} x & y & 1 \\ -y-x & x-y & 0 \\ -y & x & 0 \end{vmatrix}$

$= \dfrac{1}{2}|-xy - x^2 + xy - y^2| = \dfrac{1}{2}|-x^2 - y^2|$

$= \dfrac{1}{2}|x^2 + y^2| = \dfrac{1}{2}|z|^2$

39. $\dfrac{(1+i)x - 2i}{3+i} + \dfrac{(2-3i)y + i}{3-i} = i$

$\Rightarrow (4 + 2i)x - 6i - 2 + (9 - 7i)y + 3i - 1 = 10i$

$\Rightarrow\ (4x+9y-3)+(2x-7y-3)i=10i$

$\Rightarrow\ 4x+9y-3=0$ and $2x-7y-3=10$

On solving these two equations, we get $x=3, y=-1$

40. Given : $x+iy=\sqrt{\dfrac{a+ib}{c+id}}$

$$\Rightarrow (x+iy)^2=\frac{a+ib}{c+id} \quad ....(i)$$

Taking conjugate on both sides, we get

$$(x-iy)^2=\frac{a-ib}{c-id} \quad ....(ii)$$

On multiply (i) and (ii), we get

$$(x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$$

41. $\dfrac{1}{1-\cos\theta+2i\sin\theta}$

$$=\frac{1}{2\sin^2\theta/2+4i\sin\theta/2\cos\theta/2}=\frac{1}{2\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{(\sin\theta/2+2i\cos\theta/2)(\sin\theta/2-2i\cos\theta/2)}\right]$$

$$=\frac{1}{2\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{(\sin^2\theta/2+4\cos^2\theta/2}\right]$$

$$=\frac{1}{2\sin\theta/2}\left[\frac{2\sin\theta/2-4i\cos\theta/2}{1-\cos\theta+4+4\cos\theta}\right]$$

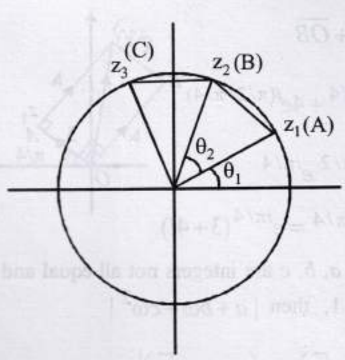
$$=\frac{2}{2\sin\theta/2}\left[\frac{2\sin\theta/2-2i\cos\theta/2}{5+3\cos\theta}\right]$$

$$=\left(\frac{1}{5+3\cos\theta}\right)+\left(\frac{-2\cot\theta/2}{5+3\cos\theta}\right)i$$

which is of the form $x+iy$.

Topic-2: Rotational Theorem, Square Root of a Complex Number, Cube Roots of Unity, Geometry of Complex Numbers, De-moiver's Theorem, Powers of Complex Numbers

1. (c)



Since, $|z_1|=|z_2|=...|z_{10}|=1$

$\theta_2=\text{arc}\,(z_1 z_2)$

$|z_2-z_1|=$ length of line AB $\le$ length of arc AB

$|z_3-z_2|=$ length of line BC $\le$ length of arc BC

$\therefore$ Sum of length of these 10 lines $\le$ Sum of length of arcs (*i.e.* 2π)

$$[\because \theta_1+\theta_2+\theta_3+...+\theta_{10}=2\pi]$$

$\therefore\ P:|z_2-z_1|+|z_3-z_2|+...+|z_1-z_{10}|\le 2\pi$

P is true.

Now, $|z_2^2-z_1^2|=|z_2-z_1||z_2+z_1|$

We know that

$|z_2+z_1|\le|z_2|+|z_1|\le 2$

$\therefore\ |z_2^2-z_1|^2+|z_3^2-z_2^2|+...+|z_1^2-z_{10}^2|\le$

$2\{|z_2-z_1|+|z_3-z_2|+...+|z_1-z_{10}|\}\le 2(2\pi)\Rightarrow Q\le 4\pi$

Q is also true.

2. (d) $S:|z-2+i|\ge\sqrt{5}$ represents boundary and outer region of circle with centre $(2,-1)$ and radius $\sqrt{5}$ units.

$z_0\in S$, such that $\dfrac{1}{|z_0-1|}$ is the maximum.

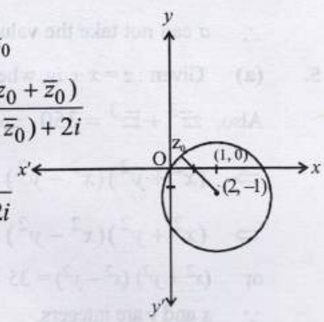
$\therefore |z_0-1|$ is minimum

$z_0\in S$ with $|z_0-1|$ as minimum will be a point on boundary of circle of region S which lies on radius of this circle, which passes through (1, 0).

$\therefore z_0, 1, 2-i$ are collinear, or (x_0, y_0), (1, 0), (2, –1) are collinear.

$\therefore$ Using slopes of paralled lines,x'

$$\frac{y_0}{x_0-1}=\frac{-1}{2-1}\Rightarrow y_0=1-x_0$$



Now, $\dfrac{4-z_0-\bar{z}_0}{z_0-z_0+2i}=\dfrac{4-(z_0+\bar{z}_0)}{(z_0-\bar{z}_0)+2i}$

$$=\frac{4-2x_0}{2iy_0+2i}=\frac{4-2x_0}{2i-2x_0i+2i}$$

$$=\frac{2(2-x_0)}{2(2-x_0)i}=\frac{1}{i}=-i$$

$$\therefore\ Arg\left(\frac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0-2i}\right)=Arg(-i)=\frac{-\pi}{2}$$

3. (c) Since, α lies on the circle $(x-x_0)^2+(y-y_0)^2=r^2$

$\therefore\ |\alpha-z_0|^2=r^2$

$\Rightarrow\ (\alpha-z_0)(\bar{\alpha}-\bar{z}_0)=r^2$

$\Rightarrow \alpha\bar{\alpha}-\alpha\bar{z}_0-\bar{\alpha}z_0+z_0\bar{z}_0=r^2$

$$\Rightarrow |\alpha|^2+|z_0|^2-\alpha\bar{z}_0-\bar{\alpha}z_0=r^2 \quad ...(i)$$

Also $\dfrac{1}{\bar{\alpha}}$ lies on the circle $(x-x_0)^2+(y-y_0)^2=4r^2$

$\therefore \left|\frac{1}{\bar{\alpha}} - z_0\right|^2 = 4r^2 \Rightarrow \left(\frac{1}{\bar{\alpha}} - z_0\right)\left(\frac{1}{\alpha} - \bar{z}_0\right) = 4r^2$

$\Rightarrow \frac{1}{\alpha\bar{\alpha}} - \frac{z_0}{\alpha} - \frac{\bar{z}_0}{\bar{\alpha}} + z_0\bar{z}_0 = 4r^2$

$\Rightarrow \frac{1}{|\alpha|^2} - \frac{z_0\bar{\alpha}}{|\alpha|^2} - \frac{\bar{z}_0\alpha}{|\alpha|^2} + |z_0|^2 = 4r^2$

$\Rightarrow 1 + |\alpha|^2 |z_0|^2 - z_0\bar{\alpha} - \bar{z}_0\alpha = 4r^2|\alpha|^2$ ...(ii)

On subtracting equation (i) from (ii), we get

$1 - |\alpha|^2 + |z_0|^2(|\alpha|^2 - 1) = r^2(4|\alpha|^2 - 1)$

or $(|\alpha|^2 - 1)(|z_0|^2 - 1) = r^2(4|\alpha|^2 - 1)$

Using $|z_0|^2 = \frac{r^2+2}{2}$, we get

$(|\alpha|^2 - 1)\frac{r^2}{2} = r^2(4|\alpha|^2 - 1)$

$\Rightarrow |\alpha|^2 - 1 = 8|\alpha|^2 - 2 \Rightarrow |\alpha| = \frac{1}{\sqrt{7}}$

4. **(d)** $\because$ $Im(z) \neq 0$

$\Rightarrow$ z is non real

and equation $z^2 + z + (1-a) = 0$ will have non real roots, if $D < 0$

$\Rightarrow 1 - 4(1-a) < 0 \Rightarrow 4a < 3 \Rightarrow a < \frac{3}{4}$

$\therefore$ a can not take the value $\frac{3}{4}$.

5. **(a)** Given : $z = x + iy$, where x and y are integer

Also, $z\bar{z}^3 + \bar{z}z^3 = 350 \Rightarrow |z|^2(\bar{z}^2 + z^2) = 350$

$\Rightarrow (x^2+y^2)(x^2-y^2) = 175$

$\Rightarrow (x^2+y^2)(x^2-y^2) = 25 \times 7$ ...(i)

or $(x^2+y^2)(x^2-y^2) = 35 \times 5$ ...(ii)

$\because$ x and y are integers,

$\therefore x^2 + y^2 = 25$ and $x^2 - y^2 = 7$ [From eq (i)]

$\Rightarrow x^2 = 16$ and $y^2 = 9$

$\Rightarrow x = \pm 4$ and $y = \pm 3$

$\therefore$ Vertices of rectangle are

(4, 3), (4, – 3), (– 4, – 3), (– 4, 3).

$\therefore$ Area of rectangle = 8 × 6 = 48 sq. units

Now from eq. (ii),

$x^2 + y^2 = 35$ and $x^2 - y^2 = 5$

$\Rightarrow x^2 = 20$, which is not possible for any integral value of x

6. **(d)** $z = \cos\theta + i\sin\theta$

$\Rightarrow z^{2m-1} = (\cos\theta + i\sin\theta)^{2m-1}$

$= \cos(2m-1)\theta + i\sin(2m-1)\theta$

$\left[\begin{array}{l}\text{By De Moivre's theorem :}\\ (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta\end{array}\right]$

$\therefore \text{Im}(z^{2m-1}) = \sin(2m-1)\theta$

$\therefore \sum_{m=1}^{15} \text{Im}(z^{2m-1}) = \sum_{m=1}^{15} \sin(2m-1)\theta$

$= \sin\theta + \sin 3\theta + \sin 5\theta + ... +$ upto 15 terms

$= \frac{\sin\left[15\left(\frac{2\theta}{2}\right)\right].\sin[\theta + 14\times\theta]}{\sin\theta}$

$\left[\begin{array}{l}\because \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ... n \text{ terms}\\ = \frac{\sin(n\beta/2).\sin[\alpha + (n-1)\beta/2]}{\sin(\beta/2)}\end{array}\right]$

$= \frac{\sin 15\theta.\sin 15\theta}{\sin\theta} = \frac{\sin 30°.\sin 30°}{\sin 2°} = \frac{1}{4\sin 2°}$

7. **(d)** The initial position of point is $Z_0 = 1 + 2i$

$\therefore Z_1 = (1+5) + (2+3)i = 6 + 5i$

Now Z_1 is moved through a distance of $\sqrt{2}$ units in the direction $\hat{i} + \hat{j}$. (i.e. by $1 + i$)

$\therefore$ It becomes $Z_1' = Z_1 + (1+i) = 7 + 6i$

Now OZ_1' is rotated through an angle $\frac{\pi}{2}$ in anticlockwise direction, therefore $Z_2 = iZ_1' = -6 + 7i$

8. **(d)** Given : $|z| = 1$ and $z \neq \pm 1$

To find the locus of $\omega = \frac{z}{1-z^2}$

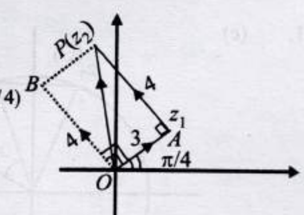
Now, $\omega = \frac{z}{1-z^2} = \frac{z}{z\bar{z} - z^2}$

$[\because |z| = 1 \Rightarrow |z|^2 = z\bar{z} = 1]$

$= \frac{1}{\bar{z} - z}$ = purely imaginary number

$\therefore$ ω must lie on y-axis.

9. **(d)** $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$



$\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$

$\Rightarrow \overrightarrow{OP} = 3e^{i\pi/4} + 4e^{i(\pi/2 + \pi/4)}$

$= 3e^{i\pi/4} + 4e^{i\pi/2}.e^{i\pi/4}$

$= 3e^{i\pi/4} + 4ie^{i\pi/4} = e^{i\pi/4}(3 + 4i)$.

10. **(b)** Given that a, b, c are integers not all equal and ω is cube root of unity $\neq 1$, then $|a + b\omega + c\omega^2|$

$= \left|a + b\left(\frac{-1+i\sqrt{3}}{2}\right) + c\left(\frac{-1-i\sqrt{3}}{2}\right)\right|$

$= \left|\left(\frac{2a-b-c}{2}\right)+i\left(\frac{b\sqrt{3}-c\sqrt{3}}{2}\right)\right|$

$= \frac{1}{2}\sqrt{(2a-b-c)^2+3(b-c)^2}$

$= \sqrt{\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]}$

R.H.S. will be minimum when $a = b = c$, but according to the question, we cannot take $a = b = c$.

$\therefore$ The minimum value is obtained when any two are zero and third is a minimum magnitude integer i.e. 1.

$\therefore b = c = 0, a = 1$; gives us the minimum value 1.

11. **(a)** In the figure, we see that.

$AB = AC = AD = 2$

$\therefore$ BCD is an arc of a circle with centre at A and radius 2. Shaded region is exterior part of this sector $ABCDA$.

$\therefore$ For any point represented by z on arc BCD we should have $|z-(-1)| = 2$

and for shaded region, $|z + 1| > 2$ ....(i)

For shaded region, we also have

$-\pi/4 < \arg(z+1) < \pi/4$

or $|\arg(z+1)| < \pi/4$ ...(ii)

From (i) and (ii), we get (a) is the correct option.

12. **(b)** $(1+\omega^2)^n = (1+\omega^4)^n$

$\Rightarrow (-\omega)^n = (1+\omega)^n = (-\omega^2)^n \Rightarrow \omega^n = 1 \Rightarrow n = 3$

13. **(a)** Given that $|z| = 1$ and $\omega = \frac{z-1}{z+1}(z \neq -1)$

Now we know that $z\bar{z} = |z|^2$

$\Rightarrow z\bar{z} = 1$ (for $|z| = 1$)

$\therefore \omega = \left(\frac{z-1}{z+1}\right)\times\frac{(\bar{z}+1)}{(\bar{z}+1)} = \frac{z\bar{z}+z-\bar{z}-1}{z\bar{z}+z+\bar{z}+1} = \frac{2iy}{2+2x}$

$[\because z\bar{z} = 1$ and taking $z = x+iy$ so that

$z+\bar{z} = 2x$ and $z-\bar{z} = 2iy]$

$\Rightarrow \text{Re}(\omega) = 0$

14. **(b)** Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1-\omega & \omega^2 \\ 1 & \omega^2 & \omega 4 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega 4 \end{vmatrix}$$

$= 3[-\omega-1-\omega] = 3(\omega^2-\omega)$

15. **(c)** $\frac{z_1-z_3}{z_2-z_3} = \frac{1-i\sqrt{3}}{2}$

$\Rightarrow \arg\left(\frac{z_1-z_3}{z_2-z_3}\right) = \arg\left(\frac{1-i\sqrt{3}}{2}\right)$

$\Rightarrow \arg(\cos(-\pi/3)+i\sin(-\pi/3))$

$\Rightarrow$ angle between (z_1-z_3) and (z_2-z_3) is $60°$.

and $\left|\frac{z_1-z_3}{z_2-z_3}\right| = \left|\frac{1-i\sqrt{3}}{2}\right|$

$\Rightarrow \left|\frac{z_1-z_3}{z_2-z_3}\right| = 1 \Rightarrow |z_1-z_3| = |z_2-z_3|$ (Imp Step)

$\Rightarrow$ The Δ with vertices z_1, z_2 and z_3 is isosceles with vertical angle $60°$. Hence rest of the two angles should also be $60°$ each.

$\Rightarrow$ Required triangle is an equilateral triangle.

16. **(d)** Let $z = (1)^{1/n} = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$

$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, k = 0, 1, 2, \ldots\ldots, n-1.$

Let $z_1 = \cos\left(\frac{2k_1\pi}{n}\right)+i\sin\left(\frac{2k_1\pi}{n}\right)$

and $z_2 = \cos\left(\frac{2k_2\pi}{n}\right)+i\sin\frac{2k_2\pi}{n}$

be the two values of z. Such that they subtend right angle at origin.

$\therefore \frac{2k_1\pi}{n} - \frac{2k_2\pi}{n} = \pm\frac{\pi}{2} \Rightarrow 4(k_1-k_2) = \pm n$

As k_1 and k_2 are integers and $k_1 \neq k_2$.

$\therefore n = 4k, k \in I$

17. **(c)** $E = 4+5(\omega)^{334}+3(\omega)^{365} = 4+5\omega+3\omega^2$

$= 1+2\omega+3(1+\omega+\omega^2) = 1+(-1+i\sqrt{3}) = i\sqrt{3}$

18. **(b)** $(1+\omega)^7 = A+B\omega$

$\Rightarrow (-\omega^2)^7 = A+B\omega$ $(\because 1+\omega+\omega^2 = 0)$

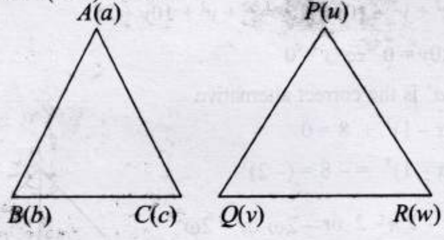
$\Rightarrow -\omega^{14} = A+B\omega$

$\Rightarrow -\omega^2 = A+B\omega$ $(\because \omega^3 = 1)$

$\Rightarrow 1+\omega = A+B\omega \Rightarrow A = 1, B = 1$

19. **(b)** Let ABC be the Δ whose vertices are represented by complex numbers a, b, c and PQR be the Δ with whose vertices are represented by complex numbers u, v, w.

Then $c = (1-r)a + rb$



$\Rightarrow c-a = r(b-a) \Rightarrow \frac{c-a}{b-a} = r$ ....(i)

$\Rightarrow w = (1-r)u + rv \Rightarrow \frac{w-u}{v-u} = r$ ....(ii)

From (i) and (ii), $\left|\frac{c-a}{b-a}\right|=\left|\frac{w-u}{v-u}\right|$

and $\arg\left(\frac{c-a}{b-a}\right)=\arg\left(\frac{w-u}{v-u}\right)$

$\Rightarrow \frac{AC}{AB}=\frac{PR}{PQ}$ and $\angle CAB=\angle RPQ$

$\Rightarrow \Delta ABC \sim \Delta PQR$

20. **(b)** If vertices of a parallelogram are z_1, z_2, z_3, z_4 then as diagonals bisect each other

$\therefore \frac{z_1+z_3}{2}=\frac{z_2+z_4}{2} \Rightarrow z_1+z_3=z_2+z_4$

21. **(b)** $|\omega|=1 \Rightarrow \left|\frac{1-iz}{z-i}\right|=1$

$\Rightarrow |1-iz|=|z-i|$

$\Rightarrow |1-i(x+iy)|=|x+iy-i|$

$\Rightarrow |(y+1)-ix|=|x+i(y-1)|$

$\Rightarrow x^2+(y+1)^2=x^2+(y-1)^2$

$\Rightarrow 4y=0 \Rightarrow y=0 \Rightarrow z$ lies on real axis

22. **(d)** $|z-4|<|z-2|$

$\Rightarrow |(x-4)+iy|<|(x-2)+iy|$

$\Rightarrow (x-4)^2+y^2<(x-2)^2+y^2$

$\Rightarrow -8x+16<-4x+4 \Rightarrow 4x-12>0$

$\Rightarrow x>3 \Rightarrow \text{Re}(z)>3$

23. **(b)** $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)=-i\left(\frac{-1}{2}+\frac{i\sqrt{3}}{2}\right)=i\omega$

$\frac{\sqrt{3}}{2}-\frac{i}{2}=i\left(\frac{-1}{2}-\frac{i\sqrt{3}}{2}\right)=i\omega^2$

$\therefore z=(-i\omega)^5+(i\omega^2)^5=-i\omega^2+i\omega$

$=i(\omega-\omega^2)=i(i\sqrt{3})=-\sqrt{3}$

$\Rightarrow \text{Re}(z)<0$ and $\text{Im}(z)=0$

24. **(a)** Since, $z=x+iy$ satisfies the equation $\left|\frac{z-5i}{z+5i}\right|=1$

$\therefore |x+iy-5i|=|x+iy+5i|$

$\Rightarrow |x+(y-5)i|=|x+(y+5)i|$

$\Rightarrow x^2+(y-5)^2=x^2+(y+5)^2$

$\Rightarrow x^2+y^2-10y+25=x^2+y^2+10y+25$

$\Rightarrow 20y=0 \Rightarrow y=0$

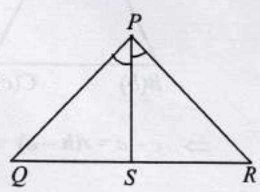
$\therefore$ '*a*' is the correct alternative.

25. **(b)** $(x-1)^3+8=0$

$\Rightarrow (x-1)^3=-8=(-2)^3$

$\Rightarrow x-1=-2$ or -2ω or $-2\omega^2$

$\Rightarrow x=-1, 1-2\omega, 1-2\omega^2$



26. **(8)** Let $z=x+iy$

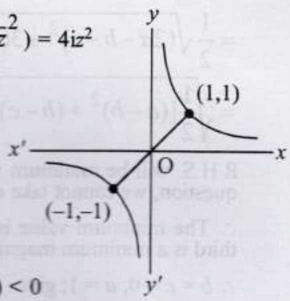
$z^4-|z|^4=4iz^2$

$\Rightarrow z^4-(z\bar{z})^2=4iz^2 \Rightarrow z^2(z^2-\bar{z}^2)=4iz^2$

$\Rightarrow z=0$ or $z^2-(\bar{z})^2=4i$

$\Rightarrow 4ixy=4i \Rightarrow xy=1$

Locus of z is a rectangular hyperbola $xy=1$



Given that $\text{Re}(z_1)>0$ and $\text{Re}(z_2)<0$

$\therefore |z_1-z_2|_{\min}=\sqrt{(1+1)^2+(1+1)^2}=\sqrt{8}$

$\Rightarrow |z_1-z_2|^2_{\min}=8$

27. **(3)** a, b, c are distinct non-zero integers

Min. value of $|a+b\omega+c\omega^2|^2$ is to be found $|a+b\omega+c\omega^2|^2$

$=\left|a+b\left(\frac{-1+i\sqrt{3}}{2}\right)+c\left(\frac{-1-i\sqrt{3}}{2}\right)\right|^2$

$=\left|\frac{1}{2}(2a-b-c)+\frac{i\sqrt{3}}{2}(b-c)\right|^2$

$=\frac{1}{4}(2a-b-c)^2+\frac{3}{4}(b-c)^2$

$=\frac{1}{4}(4a^2+b^2+c^2-4ab+2bc-4ac+3b^2+3c^2-6bc)$

$=a^2+b^2+c^2-ab-bc-ca$

$=\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$

For minimum value, let us consider $a=3, \quad b=2, \; c=1$

$\therefore$ minimum value $=\frac{1}{2}[1+1+4]=\frac{6}{2}=3$

28. *r*th term of the given series

$=r[(r+1)-\omega](r+1)-\omega^2]$

$=r[(r+1)^2-(\omega+\omega^2)(r+1)+\omega^3]$

$=r[(r+1)^2-(-1)(r+1)+1]$

$=r[(r^2+3r+3]=r^3+3r^2+3r$

$\therefore$ Sum of the given series $=\sum_{r=1}^{(n-1)}(r^3+3r^2+3r)$

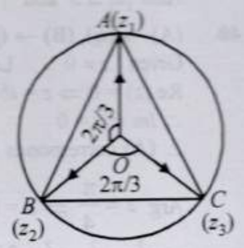
$=\frac{1}{4}(n-1)^2n^2+3.\frac{1}{6}(n-1)(n)(2n-1)+3.\frac{1}{2}(n-1)n$

$=(n-1)(n)\left[\frac{1}{4}(n-1)n+\frac{1}{2}(2n-1)+\frac{3}{2}\right]$

$=\frac{1}{4}(n-1)\,n[n^2-n+4n-2+6]$

$=\frac{1}{4}(n-1)\,n[n^2+3n+4]$

29. Let z_1, z_2, z_3 be the vertices A, B and C respectively of equilateral ΔABC, inscribed in a circle $|z| = 2$ with centre (0, 0) and rasius = 2



Given $z_1 = 1 + i\sqrt{3}$

$z_2 = e^{\frac{2\pi i}{3}} z_1$

$= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)(1 + i\sqrt{3})$

$= \frac{-1-3}{2} = -2$ and $z_3 = e^{4(\pi/3)i} z_1$

$= \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)(1 + i\sqrt{3})$

$= \left(\frac{-1-i\sqrt{3}}{2}\right)(1 + i\sqrt{3}) = \frac{-1-2i\sqrt{3}+3}{2} = 1 - i\sqrt{3}$

30. As D and m are represented by complex numbers $(1 + i)$ and $(2 - i)$ respectively

$\therefore$ $D \equiv (1,1)$ and $M \equiv (2, -1)$

We know that diagonals of rhombus bisect each other at right angles.

$\therefore$ AC passes through M and is $\perp$ to BD

$\therefore$ Eq. of AC in symmetric form can be written as

$\frac{x-2}{2/\sqrt{5}} = \frac{y+1}{1/\sqrt{5}} = r$

Now for pt. A, as

$AM = \frac{1}{2}DM = \frac{1}{2}\sqrt{(2-1)^2 + (-1-1)^2} = \sqrt{5}/2$

On putting $r = \pm\sqrt{5}/2$, we get

$\frac{x-2}{2/\sqrt{5}} = \frac{y+1}{1/\sqrt{5}} = \pm\sqrt{5}/2 \Rightarrow x = \pm 1 + 2, y = \pm\frac{1}{2} - 1$

$\Rightarrow$ $x = 3$ or $1, y = \frac{-1}{2}$ or $\frac{-3}{2}$

Therefore, point A is represented by $3 - i/2$ or $1 - (3/2)i$

31. Distance between two points represented by z_1 and z_2

$= |z_1 - z_2|$

Since $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, therefore $|z_1 - z_3| = |z_2 - z_3| = |z_1 - z_2|$

$\Rightarrow$ $|a + i| = |1 + bi| = |(a-1) + i(1-b)|$

$\Rightarrow$ $a^2 + 1 = 1 + b^2 = (a-1)^2 + (1-b)^2$

$\Rightarrow$ $a^2 = b^2 = a^2 + b^2 - 2a - 2b + 1$

$\Rightarrow$ $a = b$ ....(i)

($\because$ $a \neq -b$ because $0 < a, b < 1$)

and $b^2 - 2a - 2b + 1 = 0$

$\Rightarrow$ $a^2 - 2a - 2b + 1 = 0$ ....(ii)

$\Rightarrow$ $a^2 - 2a - 2a + 1 = 0$ $(\because a = b)$

$\Rightarrow$ $a^2 - 4a + 1 = 0$

$\therefore$ $a = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$. But $0 < a, b < 1$

$\therefore$ $a = 2 - \sqrt{3}$ $\because b = a$ $\therefore$ $b = 2 - \sqrt{3}$

32. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$= a^2|z_1|^2 + b^2|z_2|^2 - 2ab\,\mathrm{Re}(z_1\bar{z}_2) + b^2|z_1|^2 + a^2|z_2|^2 + 2ab\,\mathrm{Re}(z_1\bar{z}_2)$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

33. **(True)** $\because$ Cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2}$

$\therefore$ Vertices of triangle are

$A(1,0), B\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right), C\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$

$\Rightarrow$ $AB = BC = CA$, $\therefore$ Δ is equilateral.

34. **(False)** If z_1, z_2, z_3 are in A.P. then, $\frac{z_1 + z_3}{2} = z_2$

$\Rightarrow$ z_2 is mid pt. of line joining z_1 and z_3.

$\Rightarrow$ z_1, z_2, z_3 lie on a st. line

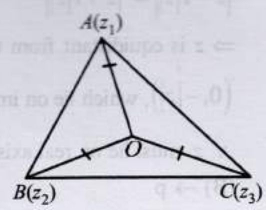
$\therefore$ Given statement is false

35. **(True)**

As $|z_1| = |z_2| = |z_3|$

$\therefore$ z_1, z_2, z_3 are equidistant from origin. Hence O is the circumcentre of ΔABC.

But ΔABC is equilateral and hence circumcentre and centriod of ΔABC coincide.



$\therefore$ Centroid of $\Delta ABC = 0$

$\Rightarrow$ $\frac{z_1 + z_2 + z_3}{3} = 0$

$\Rightarrow$ $z_1 + z_2 + z_3 = 0$

$\therefore$ Statement is true.

36. **(a, c, d)** $z = \frac{1}{a + ibt} = x + iy$

$\Rightarrow x + iy = \frac{a - ibt}{a^2 + b^2t^2} \Rightarrow x = \frac{a}{a^2 + b^2t^2}, y = \frac{-bt}{a^2 + b^2t^2}$

$\Rightarrow x^2 + y^2 = \frac{1}{a^2 + b^2t^2} = \frac{x}{a} \Rightarrow x^2 + y^2 - \frac{x}{a} = 0$

$\therefore$ Locus of z is a circle with centre $\left(\frac{1}{2a}, 0\right)$ and radius $\frac{1}{2|a|}$

irrespective of 'a' +ve or –ve

Also for b = 0, a ≠ 0, we get, y = 0

$\therefore$ locus is x-axis

and for a = 0, b ≠ 0 we get x = 0

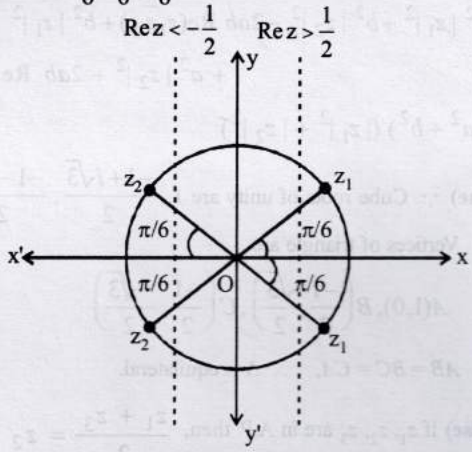
$\therefore$ locus is y-axis.

Hence, a, c, d are the correct options.

37. **(c, d)** We have $w = \frac{\sqrt{3} + i}{2} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$\Rightarrow w^n = \cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}$

$\therefore$ P contains all those points which lie on unit circle and have arguments $\frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}$ and so on.



Since, $z_1 \in P \cap H_1$ and $z_2 \in P \cap H_2$, therefore z_1 and z_2 can have possible positions as shown in the figure.

$\therefore \angle z_1Oz_2$ can be $\frac{2\pi}{3}$ or $\frac{5\pi}{6}$.

38. **(d)** We have $(1+\omega-\omega^2)^7 = (-\omega^2-\omega^2)^7$

$= (-2)^7(\omega^2)^7 = -128\omega^{14} = -128\omega^2$

39. **(A) → (q, r), B → (p), C → (p, s, t), D → (q, r, s, t)**

(A) → (q, r)

$|z-i|z|| = |z+i|z||$

$\Rightarrow z$ is equidistant from two points $(0,|z|)$ and $(0,-|z|)$, which lie on imaginary axis.

$\therefore$ z must lie on real axis $\Rightarrow \text{Im}(z)=0$. Also $|I_m(z)| \le 1$

(B) → p

Sum of distances of z from two points (–4, 0) and (4, 0) is 10 which is greater than 8.

$\therefore z$ traces an ellipse with $2a = 10$ and $2ae = 8$

$\Rightarrow e = \frac{4}{5}$

(C) → (p, s, t)

Let $\omega = 2(\cos\theta + i\sin\theta)$, then

$$z = \omega - \frac{1}{\omega} = 2(\cos\theta + i\sin\theta) - \frac{1}{2}(\cos\theta - i\sin\theta)$$

$\Rightarrow x + iy = \frac{3}{2}\cos\theta + i\frac{5}{2}\sin\theta$

Here, $|z| = \sqrt{\frac{9+25}{4}} = \sqrt{\frac{34}{4}} \le 3$ and $|R_e(z)| \le 2$

Also $x = \frac{3}{2}\cos\theta, y = \frac{5}{2}\sin\theta \Rightarrow \frac{4x^2}{9} + \frac{4y^2}{25} = 1$

Which is an ellipse with $e = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

(D) → (q, r, s, t)

Let $\omega = \cos\theta + i\sin\theta$ then $z = 2\cos\theta \Rightarrow \text{Im}\, z = 0$

Also $|z| \le 3$ and $|\text{Im}(z)| \le 1, |R_e(z)| \le 2$

40. **(A) → (q), (B) → (p)**

Given : $z \ne 0$ Let $z = a + ib$

Re $(z) = 0 \Rightarrow z = ib \Rightarrow z^2 = -b^2$

$\therefore Im\,(z)^2 = 0$

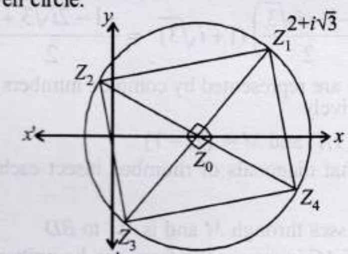
$\therefore$ (A) corresponds to (q)

Arg $z = \frac{\pi}{4} \Rightarrow a = b \Rightarrow z = a + ia$

$\Rightarrow z^2 = a^2 - a^2 + 2ia^2 \Rightarrow z^2 = 2ia^2 \Rightarrow$ Re $(z)^2 = 0$

$\therefore$ (B) corresponds to (p).

41. The given circle is $|z-1| = \sqrt{2}$, where z_0=1 is the centre and $\sqrt{2}$ is radius of circle. z_1 is one of the vertex of square inscribed in the given circle.



Clearly z_2 can be obtained by rotating z_1 by an angle 90° in anticlockwise direction, about centre z_0

Thus, $z_2 - z_0 = (z_1 - z_0)\, e^{i\pi/2}$

or $z_2 - 1 = (2 + i\sqrt{3} - 1)i \Rightarrow z_2 = i - \sqrt{3} + 1$

$z_2 = (1-\sqrt{3}) + i$

Again rotating z_2 by 90° about z_0, we get

$z_3 - z_0 = (z_2 - z_0)\, i$

$\Rightarrow z_3 - 1 = [(1-\sqrt{3}) + i - 1]\, i = -\sqrt{3}i - 1$

$\Rightarrow z_3 = -i\sqrt{3}$

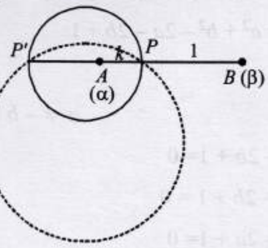
And similarly $1 = (-i\sqrt{3} - 1)\, i = \sqrt{3} - i$

$\Rightarrow z_4 = (\sqrt{3}+1) - i$

Hence, remaining vertices are

$(1-\sqrt{3}) + i, -i\sqrt{3}, (\sqrt{3}+1) - i$

42. Given : $\left|\frac{z-\alpha}{z-\beta}\right| = k \Rightarrow |z-\alpha| = k|z-\beta|$



Let pt. A represents complex number α and B that of β, and P represents z. then $|z-\alpha| = k|z-\beta|$

$\Rightarrow$ z is the complex number whose distance from A is k times its distance from B.

i.e. $PA = k\,PB$

$\Rightarrow P$ divides AB in the ratio $k:1$ internally or externally (at P').

Then $P=\left(\frac{k\beta+\alpha}{k+1}\right)$ and $P'=\left(\frac{k\beta-\alpha}{k-1}\right)$

Now through PP' a number of circles can pass, but with given data we can find radius and centre of that circle for which PP' is diameter.

$\therefore$ Centre = mid. point of PP' $=\left(\frac{\frac{k\beta+\alpha}{k+1}+\frac{k\beta-\alpha}{k-1}}{2}\right)$

$=\frac{k^2\beta+k\alpha-k\beta-\alpha+k^2\beta-k\alpha+k\beta-\alpha}{2(k^2-1)}$

$=\frac{k^2\beta-\alpha}{k^2-1}=\frac{\alpha-k^2\beta}{1-k^2}$. Also radius $=\frac{1}{2}|PP'|$

$=\frac{1}{2}\left|\frac{k\beta+\alpha}{k+1}-\frac{k\beta-\alpha}{k-1}\right|$

$=\frac{1}{2}\left|\frac{k^2\beta+k\alpha-k\beta-\alpha-k^2\beta+k\alpha-k\beta+\alpha}{k^2-1}\right|=\frac{k|\alpha-\beta|}{|1-k^2|}$

43. Let us consider, $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r|<2$

$\Rightarrow a_1 z + a_2 z^2 + a_3 z^3 + ... + a_n z^n = 1$

$\Rightarrow |a_1 z + a_2 z^2 + a_3 z^3 + ... + a_n z^n| = 1$ ....(i)

But we know that $|z_1+z_2|\le|z_1|+|z_2|$

$\therefore$ Using its generalised form, we get
$|a_1 z + a_2 z^2 + a_3 z^3 + ... + a_n z^n|$
$\le |a_1 z| + |a_2 z^2| + ... + |a_n z^n|$

$\Rightarrow 1 \le |a_1||z| + |a_2||z^2| + |a_3||z^3| + ... + |a_n||z^n|$ [using eq^n (i)]

But given that $|a_r|<2\ \forall r=1,...,n$

$\therefore 1<2[|z|+|z|^2+|z|^3+...+|z|^n]$

$[\because |z^n|=|z|^n]$

$\Rightarrow 1<2\left[\frac{|z|(1-|z|^n)}{1-|z|}\right]\Rightarrow 2\left[\frac{|z|-|z|^{n+1}}{1-|z|}\right]>1$

$\Rightarrow 2[|z|-|z|^{n+1}]>1-|z|$ $(\because 1-|z|>0$ as $|z|<1/3)$

$\Rightarrow [|z|-|z|^{n+1}]>\frac{1}{2}-\frac{1}{2}|z|$

$\Rightarrow \frac{3}{2}|z|>\frac{1}{2}+|z|^{n+1}$

$\Rightarrow |z|>\frac{1}{3}+\frac{2}{3}|z|^{n+1}\Rightarrow |z|>\frac{1}{3}$

which is a contradiction as given that $|z|<\frac{1}{3}$

$\therefore$ There exist no such complex number.

44. The given equation can be written as
$(z^p-1)(z^q-1)=0$

$\therefore z=(1)^{1/p}$ or $(1)^{1/q}$ ....(i)

where p and q are distinct prime numbers.

Hence both the equations will have distinct roots and as $z\ne1$, both will not be simultaneously zero for any value of z given by equations in (i)

Also $1+\alpha+\alpha^2+...+\alpha^{p-1}=\frac{1-\alpha^p}{1-\alpha}=0\ (\alpha\ne1)$

or $1+\alpha+\alpha^2+...+\alpha^p=\frac{1-\alpha^q}{1-\alpha}=0\ (\alpha\ne1)$

Because of (i) either $\alpha^p=1$ and if $\alpha^q=1$ but not both simultaneously as p and q are distinct primes.

45. $|z|^2\omega-|\omega|^2 z=z-\omega$ ..... (i)

$z\bar{z}\omega-\omega\bar{\omega}z=z-\omega$

$\Rightarrow z\omega(\bar{z}-\bar{\omega})=z-\omega.$

Taking modulus, $|z\cdot\omega||\bar{z}-\bar{\omega}|=z-\omega$

$|z\omega||z-\omega|=z-\omega.$

$\Rightarrow |z-\omega|(|z\omega|-1)=0$

If $|z-\omega|=0$ then $z-\omega=0$ $\therefore z=\omega.$

If $|z\omega|-1=0$ then $z\omega=1$ $\therefore |z|=\frac{1}{|\omega|}=r$ (say)

Let $z=re^{i\theta},\ \omega=\frac{1}{r}e^{i\phi}$

From (i) $r^2\left(\frac{1}{r}e^{i\phi}\right)-\frac{1}{r^2}(re^{i\theta})=re^{i\theta}-\frac{1}{r}e^{i\phi}$

$\therefore \left(r+\frac{1}{r}\right)e^{i\phi}=\left(r+\frac{1}{r}\right)e^{i\theta}$

$e^{i\phi}=e^{i\theta}\Rightarrow\theta=\phi$

$\therefore z\bar{\omega}=(re^{i\theta})\left(\frac{1}{r}e^{-i\theta}\right)=1$ $\therefore z=\omega$ or $z\bar{\omega}=1$

46. $z^2+pz+q=0$

$z_1+z_2=-p,\ z_1z_2=q$

By rotation through α in anticlockwise direction,

$z_2=z_1e^{i\alpha}$

$\frac{z_2}{z_1}=\frac{e^{i\alpha}}{1}=\frac{\cos\alpha+i\sin\alpha}{1}$

Add 1 in both sides to get $z_1+z_2=-p$

$\therefore \frac{z_1+z_2}{z_1}=\frac{1+\cos\alpha+i\sin\alpha}{1}=2\cos\frac{\alpha}{2}\left[\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right]$

$\Rightarrow \frac{(z_2+z_1)}{z_1}=2\cos\frac{\alpha}{2}e^{i\alpha/2}$

On squaring, $(z_2+z_1)^2=4\cos^2(\alpha/2)z_1^2.e^{i\alpha}$

$=4\cos^2\frac{\alpha}{2}z_1^2.\frac{z_2}{z_1}=4\cos^2\frac{\alpha}{2}z_1z_2$

$\Rightarrow p^2=4q\cos^2\frac{\alpha}{2}$

47. Let $z=x+iy$ then $\bar{z}=iz^2$

$\Rightarrow x-iy=i(x^2-y^2+2ixy)$

$\Rightarrow x-iy=i(x^2-y^2)-2xy$

$\Rightarrow x(1+2y)=0;\ x^2-y^2+y=0$

$\Rightarrow x=0$ or $y=-\frac{1}{2}$ $\Rightarrow x=0,\ y=0,\ 1$

or $y=-\frac{1}{2},\ x=\pm\frac{\sqrt{3}}{2}$

For non zero complex number z,

$x=0, y=1;\ x=\frac{\sqrt{3}}{2},\ y=-\frac{1}{2};\ x=\frac{-\sqrt{3}}{2},\ y=-\frac{1}{2}$

$\therefore \quad z=i, \frac{\sqrt{3}}{2}-\frac{i}{2}, -\frac{\sqrt{3}}{2}-\frac{i}{2}$

48. Let $z=r_1(\cos\theta_1+i\sin\theta_1)$ and $w=r_2(\cos\theta_2+i\sin\theta_2)$

We have, $|z|=r_1, |w|=r_2, \arg(z)=\theta_1$ and $\arg(w)=\theta_2$

Given, $|z|\le 1, |w|<1$

$\Rightarrow \quad r_1\le 1$ and $r_2\le 1$

Now, $z-w=(r_1\cos\theta_1-r_2\cos\theta_2)+i(r_1\sin\theta_1-r_2\sin\theta_2)$

$\Rightarrow \quad |z-w|^2=(r_1\cos\theta_1-r_2\cos\theta_2)^2+(r_1\sin\theta_1-r_2\sin\theta_2)^2$

$=r_1^2\cos^2\theta_1+r_2^2\cos^2\theta_2-2r_1r_2\cos\theta_1\cos\theta_2$

$+r_1^2\sin^2\theta_1+r_2^2\sin^2\theta_2-2r_1r_2\sin\theta_1\sin\theta_2$

$=r_1^2(\cos^2\theta_1+\sin^2\theta_1)+r_2^2(\cos^2\theta_2+\sin^2\theta_2)$

$-2r_1r_2(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)$

$=r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)$

$=(r_1-r_2)^2+2r_1r_2[1-\cos(\theta_1-\theta_2)]$

$=(r_1-r_2)^2+4r_1r_2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)$

$\le|r_1-r_2|^2+4\left|\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right|^2 \qquad [\because r_1, r_2\le 1]$

and $|\sin\theta|\le|\theta|, \forall\,\theta\in R$

Therefore, $|z-w|^2\le|r_1-r_2|^2+4\left|\frac{\theta_1-\theta_2}{2}\right|^2$

$\le|r_1-r_2|^2+|\theta_1-\theta_2|^2$

$\Rightarrow \quad |z-w|^2\le(|z|-|w|)^2+(\arg z-\arg w)^2$

49. Dividing through out by i and knowing that $\frac{1}{i}=-i$, we get

$z^3-iz^2+iz+1=0$

$\Rightarrow \quad z^2(z-i)+i(z-i)=0 \quad$ as $\ 1=-i^2$

$\Rightarrow \quad (z-i)(z^2+i)=0 \therefore z=i \ $ or $\ z^2=-i$

$\therefore \quad |z|=|i|=1$ or $|z^2|=|z|^2=|-i|=1 \Rightarrow |z|=1$

Hence, in either case $|z|=1$

50. $1, a_1, a_2, \ldots, a_{n-1}$ are the n roots of unity. Therefore they are roots of eq. $x^n-1=0$

Therefore by factor theorem,

$x^n-1=(x-1)(x-a_1)(x-a_2)\ldots(x-a_{n-1})$ ....(i)

$\Rightarrow \quad \frac{x^n-1}{x-1}=(x-a_1)(x-a_2)\ldots(x-a_{n-1})$ ....(ii)

On differentiating both sides of eq. (i), we get

$nx^{n-1}=(x-a_1)(x-a_2)\ldots(x-a_{n-1})+(x-1)(x-a_2)$

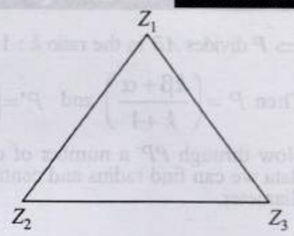
$\ldots(x-a_{n-1})+\ldots+(x-1)(x-a_1)\ldots(x-a_{n-2})$

For $x=1$, we get $n=(1-a_1)(1-a_2)\ldots(1-a_{n-1})$

[Since the terms except first, contain $(x-1)$ and hence become zero for $x=1$]

51. We know that if z_1, z_2, z_3 are vertices of an equilateral triangle, then

$\frac{z_1-z_2}{z_3-z_2}=\frac{z_3-z_1}{z_2-z_1}$

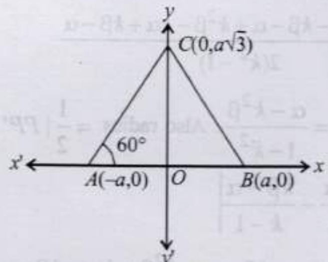


Here $z_3=0$,

$\therefore \quad \frac{z_1-z_2}{-z_2}=\frac{-z_1}{z_2-z_1}$

$\Rightarrow -z_1^2-z_2^2+2z_1z_2=z_1z_2, \quad \therefore \quad z_1^2+z_2^2-z_1z_2=0.$

52.



Let us consider the equilateral triangle with each side of length $2a$ and having two of its vertices $A(-a,0)$ and $B(a,0)$ on x-axis, then third vertex C will clearly lie on y-axis such that $OC=2a\sin 60°=a\sqrt{3}$, $\therefore C=(0,a\sqrt{3})$.

Now if A, B and C are represented by complex number z_1, z_2, z_3 then $z_1=-a$; $z_2=a$; $z_3=a\sqrt{3}\,i$

Since in an equilateral triangle, centriod and circumcentre coincide,

$\therefore$ Circumcentre, $z_0=\frac{z_1+z_2+z_3}{3}$

$\Rightarrow \quad z_0=\frac{-a+a+a\sqrt{3}\,i}{3}=\frac{ia}{\sqrt{3}}$

Now, $z_1^2+z_2^2+z_3^2=a^2+a^2-3a^2=-a^2$

and $3z_0^2=(ia)^2=-a^2$

$\therefore$ Clearly $3z_0^2=z_1^2+z_2^2+z_3^2$

53. Since, β and γ are the complex cube roots of unity therefore, we can suppose $\beta=\omega$ and $\gamma=\omega^2$ so that $\omega+\omega^2+1=0$ and $\omega^3=1$.

Then $xyz=(a+b)(a\omega^2+b\omega)(a\omega+b\omega^2)$

$=(a+b)(a^2\omega^3+ab\omega^4+ab\omega^2+b^2\omega^3)$

$=(a+b)(a^2+ab\omega+ab\omega^2+b^2)$ (using $\omega^3=1$)

$=(a+b)(a^2+ab(\omega+\omega^2)+b^2)$

$=(a+b)(a^2-ab+b^2)=a^3+b^3$

Topic-3: Solutions of Quadratic Equations, Sum and Product of Roots, Nature of Roots, Relation Between Roots and Co-efficients, Formation of an Equation with Given Roots

1. **(d)** Consider the quadratic polynomials in the form of equation

$x^2+20x-2020=0$ ...(i)

$x^2-20x+2020=0$ ...(ii)

Since, a and b are roots of the equation(i), then

$a+b=-20, ab=-2020$

∵ c and d are the roots of the equation (ii), then
$c + d = 20, cd = 2020$
Now,
$ac(a-c) + ad(a-d) + bc(b-c) + bd(b-d)$
$= a^2c - ac^2 + a^2d - ad^2 + b^2c - bc^2 + b^2d - bd^2$
$= a^2(c+d) + b^2(c+d) - c^2(a+b) - d^2(a+b)$
$= (c+d)(a^2+b^2) - (a+b)(c^2+d^2)$
$= (c+d)((a+b)^2 - 2ab) - (a+b)((c+d)^2 - 2cd)$
$= 20[(20)^2 + 4040] + 20[(20)^2 - 4040]$
$= 20 \times 800 = 16000$

2. (c) $x^2 - 2x\sec\theta + 1 = 0 \Rightarrow x = \sec\theta \pm \tan\theta$
and $x^2 + 2x\tan\theta - 1 = 0 \Rightarrow x = -\tan\theta \pm \sec\theta$

$$\because -\frac{\pi}{6} < \theta < -\frac{\pi}{12}$$

$$\Rightarrow \sec\frac{\pi}{6} > \sec\theta > \sec\frac{\pi}{12}$$

and $-\tan\frac{\pi}{6} < \tan\theta < -\frac{\tan\pi}{12}$

Also $\tan\frac{\pi}{12} < -\tan\theta < \tan\frac{\pi}{6}$

Since, α_1, β_1 are roots of $x^2 - 2x\sec\theta + 1 = 0$
and $\alpha_1 > \beta_1$
$\therefore \alpha_1 = \sec\theta - \tan\theta$ and $\beta_1 = \sec\theta + \tan\theta$
Since, α_2, β_2 are roots of $x^2 + 2x\tan\theta - 1 = 0$
and $\alpha_2 > \beta_2$
$\therefore \alpha_2 = -\tan\theta + \sec\theta, \beta_2 = -\tan\theta - \sec\theta$
$\therefore \alpha_1 + \beta_2 = \sec\theta - \tan\theta - \tan\theta - \sec\theta = -2\tan\theta$

3. (d) Quadratic equation with real coefficients and purely imaginary roots can be considered as
$p(x) = x^2 + a = 0$ where $a > 0$ and $a \in R$
The $p[p(x)] = 0 \Rightarrow (x^2 + a)^2 + a = 0$

$$\Rightarrow x^4 + 2ax^2 + (a^2 + a) = 0$$

$$\Rightarrow x^2 = \frac{-2a \pm \sqrt{4a^2 - 4a^2 - 4a}}{2}$$

$$\Rightarrow x^2 = -a \pm \sqrt{a}\, i$$

$$\Rightarrow x = \sqrt{-a \pm \sqrt{a}\, i} = \alpha \pm i\beta, \text{ where } \alpha, \beta \neq 0$$

$\therefore p[p(x)] = 0$, has complex roots which are neither purely real nor purely imaginary.

4. (c) Consider $-3(x - [x])^2 + 2[x - [x]) + a^2 = 0$
$\Rightarrow 3\{x\}^2 - 2\{x\} - a^2 = 0$ $(\because x - [x] = \{x\})$

$$\Rightarrow 3\left(\{x\}^2 - \frac{2}{3}\{x\}\right) = a^2, a \neq 0$$

$$\Rightarrow a^2 = 3\left(\{x\} - \frac{1}{3}\right)^2 - \frac{1}{3} \quad (\because 0 \le \{x\} < 1)$$

$$\frac{-1}{3} \le \{x\} - \frac{1}{3} < \frac{2}{3}; \; 0 \le 3\left(\{x\}\frac{-1}{3}\right) < \frac{4}{3}$$

$$-\frac{1}{3} \le 3\left(\{x\}\frac{-1}{3}\right) - \frac{1}{3} < 1$$

For non-integral solution $0 < a^2 < 1$
$\Rightarrow a \in (-1, 0) \cup (0, 1)$

5. (c) $\because \alpha, \beta$ are the roots of $x^2 - 6x - 2 = 0$
$\therefore \alpha^2 - 6\alpha - 2 = 0$
$\Rightarrow \alpha^{10} - 6\alpha^9 - 2\alpha^8 = 0$
$\Rightarrow \alpha^{10} - 2\alpha^8 = 6\alpha^9$ ...(i)
Similarly $\beta^{10} - 2\beta^8 = 6\beta^9$ ...(ii)
On subtracting (ii) from (i),
$\alpha^{10} - \beta^{10} - 2(\alpha^8 - \beta^8) = 6(\alpha^9 - \beta^9)$

$$\Rightarrow a_{10} - 2a_8 = 6a_9 \Rightarrow \frac{a_{10} - 2a_8}{2a_9} = 3$$

6. (c) Given : $(2x)^{\ell n2} = (3y)^{\ell n3}$
$\Rightarrow \ell n2.\ \ell n2x = \ell n3.\ \ell n3y$
$\Rightarrow \ell n2.\ \ell n2x = \ell n3.\ (\ell n3 + \ell ny)$ ...(i)
Also given : $3^{\ell nx} = 2^{\ell ny}$

$$\Rightarrow \ell nx.\ \ell n3 = \ell ny.\ \ell n2 \Rightarrow \ell ny = \frac{\ell nx.\ell n3}{\ell n2} \quad \text{...(ii)}$$

From equation (i) and (ii), we get

$$\ell n2.\ \ell n2x = \ell n3\left[\ell n3 + \frac{\ell nx.\ell n3}{\ell n2}\right]$$

$\Rightarrow (\ell n2)^2\ \ell n2x = (\ell n3)^2\ \ell n2 + (\ell n3)^2\ \ell nx$
$\Rightarrow (\ell n2)^2\ \ell n2x = (\ell n3)^2(\ell n2 + \ell nx)$
$\Rightarrow (\ell n2)^2\ \ell n2x - (\ell n3)^2\ \ell n2x = 0$
$\Rightarrow [(\ell n2)^2 - (\ell n3)^2]\ \ell n2x = 0 \Rightarrow \ell n2x = 0$

$$\Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$$

7. (b) Given : $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$

$$\Rightarrow (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = q$$

$$\Rightarrow -p^3 - 3\alpha\beta(-p) = q \Rightarrow \alpha\beta = \frac{p^3 + q}{3p}$$

Now for required quadratic equation,

$$\text{Sum of roots} = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{p^2 - 2\left(\frac{p^3+q}{3p}\right)}{\frac{p^3+q}{3p}} = \frac{p^3 - 2q}{p^3 + q}$$

and Product of roots $= \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 1$

$$\therefore \text{ Required equation is } x^2 - \left(\frac{p^3 - 2q}{p^3 + q}\right)x + 1 = 0$$

$$\Rightarrow (p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$$

8. (d) Since α and β are the roots of $x^2 - px + r = 0$
$\therefore \alpha + \beta = p$ ....(i)
and $\alpha\beta = r$ ....(ii)

Also $\frac{\alpha}{2}$ and 2β are the roots of $x^2 - qx + r = 0$

$\therefore \quad \frac{\alpha}{2} + 2\beta = q \Rightarrow \alpha + 4\beta = 2q$ ....(iii)

Solving (i) and (iii) for α and β ,we get

$\beta = \frac{1}{3}(2q - p)$ and $\alpha = \frac{2}{3}(2q - q)$

On substituting the values of α and β, in equation (ii),

we get $\frac{2}{9}(2p - q)(2q - p) = r.$

9. (a) $\because$ a, b, c are sides of a triangle and $a \neq b \neq c$

$\therefore \quad |a-b| < |c| \Rightarrow a^2 + b^2 - 2ab < c^2$ ....(i)

Similarly,

$b^2 + c^2 - 2bc < a^2$ ....(ii) ; $c^2 + a^2 - 2ca < b^2$ ....(iii)

On adding, (i), (ii) and (iii) we get

$a^2 + b^2 + c^2 < 2(ab + bc + ca)$

$\Rightarrow \quad \frac{a^2 + b^2 + c^2}{ab + bc + ca} < 2$ ....(iv)

$\because$ Roots of the given equation are real

$\therefore \quad (a + b + c)^2 - 3\lambda(ab + bc + ca) \geq 0$

$\Rightarrow \quad \frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq 3\lambda - 2$ ....(v)

From (iv) and (v), $3\lambda - 2 < 2 \Rightarrow \lambda < \frac{4}{3}$.

10. (a) $x^2 + px + q = 0$

Let roots be α and α^2, then

$\alpha + \alpha^2 = -p, \alpha\alpha^2 = q \Rightarrow \alpha = q^{1/3}$

$\therefore \quad (q)^{1/3} + (q^{1/3})^2 = -p$

On taking cube on both sides, we get

$q + q^2 + 3q(q^{1/3} + q^{2/3}) = -p^3$

$\Rightarrow \quad q + q^2 - 3pq = -p^3 \Rightarrow p^3 + q^2 - q(3p - 1) = 0$

11. (c) Let α, α^2 be the roots of $3x^2 + px + 3 = 0$

$\therefore \quad \alpha + \alpha^2 = -p/3$ and $\alpha^3 = 1$

$\Rightarrow \quad (\alpha - 1)(\alpha^2 + \alpha + 1) = 0 \quad \Rightarrow \alpha = 1$ or $\alpha^2 + \alpha = -1$

If α = 1, then $p = -6$, which is not possible as $p > 0$

If $\alpha^2 + \alpha = -1 \Rightarrow -p/3 = -1 \Rightarrow p = 3.$

12. (d) Given : $(x - a)(x - b) - 1 = 0, b > a.$

or $x^2 - (a + b)x + (ab - 1) = 0$

Let $f(x) = x^2 - (a + b)x + (ab - 1)$

$D = (a + b)^2 - 4(ab - 1)$

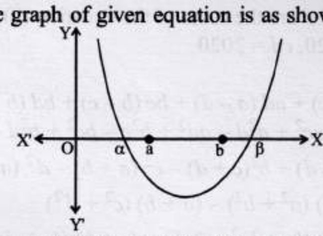
$= (a - b)^2 + 1 > 0$

Since coeff. of x^2 i.e. $1 > 0$, $\therefore f(x)$ represents upward parabola, intersecting x-axis at two points corresponding to two real roots, D being +ve. Also $f(a) = f(b) = -1$

$\Rightarrow$ curve is below x-axis at a and b

$\therefore \quad a$ and b both lie between the roots.

Therefore, the graph of given equation is as shown.



It is clear from graph, that one root of the equation lies in $(-\infty, a)$ and other in (b, ∞).

13. (b) Given : $c < 0 < b$ and $\alpha + \beta = -b$ ....(i)

$\alpha\beta = c$ ....(ii)

From (ii), $c < 0 \Rightarrow \alpha\beta < 0 \Rightarrow$ Either α is -ve or β is -ve and second quantity is positive.

From (i), $b > 0 \Rightarrow -b < 0 \Rightarrow \alpha + \beta < 0$

$\Rightarrow$ the sum is negative

$\Rightarrow$ (Modules of nengative quantity) > (Modulus of positive quantity)

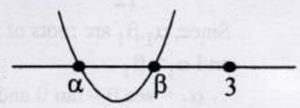
But given $\alpha < \beta$. Therefore, it is clear that α is negative and β is positive and modulus of α is greater than modulus of β

$\Rightarrow \quad \alpha < 0 < \beta < |\alpha|$

14. (a) If both roots of a quadratic equation $ax^2 + bx + c = 0$ are less than k, then

$af(k) > 0, D \geq 0, \alpha + \beta < 2k.$

$f(x) = x^2 - 2ax + a^2 + a - 3 = 0,$



$f(3) > 0, \alpha + \beta < 6, D \geq 0.$

$\Rightarrow \quad a^2 - 5a + 6 > 0, a < 3, -4a + 12 \geq 0$

$\Rightarrow \quad a < 2$ or $a > 3, a < 3, a < 3 \Rightarrow a < 2.$

15. (c) For the equation $px^2 + qx + 1 = 0$ to have real roots

$D \geq 0 \Rightarrow q^2 \geq 4p$

If $p = 1$ then $q^2 \geq 4 \Rightarrow q = 2, 3, 4$

If $p = 2$ then $q^2 \geq 8 \Rightarrow q = 3, 4$

If $p = 3$ then $q^2 \geq 12 \Rightarrow q = 4$

If $p = 4$ then $q^2 \geq 16 \Rightarrow q = 4$

$\therefore$ Number of required equations = 7

16. (c) α, β are roots of the equation $(x - a)(x - b) = c, c \neq 0$

$\therefore \quad (x - a)(x - b) - c = (x - \alpha)(x - \beta)$

$\Rightarrow \quad (x - \alpha)(x - \beta) + c = (x - a)(x - b)$

$\Rightarrow$ Roots of $(x - \alpha)(x - \beta) + c = 0$ are a and b.

17. (d) If $f(\alpha)$ and $f(\beta)$ are of opposite signs then there must lie a value γ between α and β such that $f(\gamma) = 0.$

a, b, c are real numbers and $a \neq 0.$

Since α is a root of $a^2x^2 + bx + c = 0$

$\therefore \ a^2\alpha^2 + b\alpha + c = 0$ ....(i)

Also β is a root of $a^2x^2 - bx - c = 0$

$\therefore \ a^2\beta^2 - b\beta - c = 0$ .... (ii)

Now, let $f(x) = a^2x^2 + 2bx + 2c$

Then $f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = a^2\alpha^2 + 2(b\alpha + c)$

$= a^2\alpha^2 + 2(-a^2\alpha^2)$ [using eq. (i)]

$= -a^2\alpha^2.$

and $f(\beta) = a^2\beta^2 + 2b\beta + 2c = a^2\beta^2 + 2(b\beta + c)$

$= a^2\beta^2 + 2(a^2\beta^2)$ [using eq. (ii)]

$= 3a^2\beta^2 > 0.$

Since $f(\alpha)$ and $f(\beta)$ are of opposite signs and γ is a root of equation $f(x) = 0$

$\therefore$ γ must lie between α and β

$\Rightarrow \alpha < \gamma < \beta$.

18. **(a)** Given : $x - \dfrac{2}{x-1} = 1 - \dfrac{2}{x-1}$

Clearly $x \neq 1$ for the given equation to be defined. If $x \neq 1$, we can cancel the common term $\dfrac{-2}{x-1}$ on both sides to get $x = 1$, but it is not possible. So given equation has no roots.

19. **(c)** Since, $(x^2 + px + 1)$ is a factor of $ax^3 + bx + c$, hence we can assume that zeros of $x^2 + px + 1$ are α, β and that of $ax^3 + bx + c$ be α, β, γ

$\therefore$ $\alpha + \beta = -p$ .... (i)

$\alpha\beta = 1$ .... (ii)

and $\alpha + \beta + \gamma = 0$ .... (iii)

$\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{b}{a}$ .... (iv)

$\alpha\beta\gamma = \dfrac{-c}{a}$ .... (v)

On solving (ii) and (v), we get $\gamma = -c/a$.

On solving (i) and (iii), we get $\gamma = p$

$\therefore$ $p = \gamma = -c/a$

Using equations (i), (ii) and (iv), we get

$1 + \gamma(-p) = \dfrac{b}{a}$

$\Rightarrow 1 + \left(-\dfrac{c}{a}\right)\left(\dfrac{c}{a}\right) = \dfrac{b}{a}$ $(\because \gamma = p = -c/a)$

$\Rightarrow 1 - \dfrac{c^2}{a^2} = \dfrac{b}{a} \Rightarrow a^2 - c^2 = ab$

20. **(b)** Given :

$(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b) = 0$

$\Rightarrow 3x^2 - 2(a+b+c)x + (ab+bc+ca) = 0$

$D = 4(a+b+c)^2 - 12(ab+bc+ca)$

$= 4[a^2 + b^2 + c^2 - ab - bc - ca]$

$= 2[(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0 \quad \forall\, a, b, c$

$\therefore$ Roots of given equation are always real.

21. **(c)** ℓ, m, n are real, $\ell \neq m$

Given : $(\ell - m)x^2 - 5(\ell + m)x - 2(\ell - m) = 0$

$D = 25(\ell + m)^2 + 8(\ell - m)^2 > 0, \ell, m \in R$

$\therefore$ Roots are real and unequal.

22. (1) Taking log with base 5 on the both sides

$(16(\log_5 x)^3 - 68(\log_5 x))(\log_5 x) = -16$

Let $(\log_5 x) = t$

$16t^4 - 68t^2 + 16 = 0$

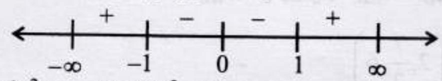
$\Rightarrow 4t^4 - 16t^2 - t^2 + 4 = 0$

$\Rightarrow (4t^2 - 1)(t^2 - 4) = 0$

or $t = \pm\dfrac{1}{2}, \pm 2$

So $\log_5 x = \pm\dfrac{1}{2}$, or $\pm 2 \Rightarrow x = 5^{\frac{1}{2}}, 5^{\frac{-1}{2}}, 5^2, 5^{-2}$

Product $= 5^{\frac{1}{2}}.5^{\frac{-1}{2}}.5^2.5^{-2} = 1$

23. **(4)** 

$3x^2 + x - 1 = 4|x^2 - 1|$

Case 1: If $x \in [-1, 1]$,

$3x^2 + x - 1 = -4x^2 + 4$

$\Rightarrow 7x^2 + x - 5 = 0 \because D = 141 > 0 \therefore$ Equation has two roots

Case 2: If $x \in (-\infty, -1] \cup [1, \infty)$

$3x^2 + x - 1 = 4x^2 - 4$

$\Rightarrow x^2 - x - 3 = 0 \because D = 13 > 0$

$\therefore$ Equation has two roots

So, total 4 roots.

24. **(2)** The given equation is

$x^2 - 8kx + 16(k^2 - k + 1) = 0$

$\because$ Both the roots are real and distinct.

$\therefore$ $D > 0 \Rightarrow (8k)^2 - 4 \times 16(k^2 - k + 1) > 0$

$\Rightarrow$ $k > 1$ ...(i)

$\because$ Both the roots are greater than or equal to 4

$\therefore$ $\alpha + \beta > 8$ and $f(4) \geq 0$

$\Rightarrow$ $k > 1$ ...(ii)

and $16 - 32k + 16(k^2 - k + 1) \geq 0$

$\Rightarrow$ $k^2 - 3k + 2 \geq 0 \Rightarrow (k-1)(k-2) \geq 0$

$\Rightarrow$ $k \in (-\infty, 1] \cup [2, \infty)$ ...(iii)

Combining (i), (ii) and (iii), we get $k \geq 2$

$\therefore$ Smallest value of $k = 2$.

25. The given equation : $x^2 - 3kx + 2e^{2\ln k} - 1 = 0$

$\Rightarrow x^2 - 3kx + (2k^2 - 1) = 0$

Now, product of roots $= 2k^2 - 1$

$\therefore 2k^2 - 1 = 7 \Rightarrow k^2 = 4 \Rightarrow k = 2, -2$

For real roots, $D \geq 0$

$\Rightarrow 9k^2 - 4(2k^2 - 1) \geq 0 \Rightarrow k^2 + 4 \geq 0$,

which is true for all k. Thus $k = 2, -2$

But for $k = -2$, $\ln k$ is not defined

We reject $k = -2$, we get $k = 2$.

26. Since, p and q are real and one root is $2 + i\sqrt{3}$, therefore other root should be $2 - i\sqrt{3}$

$\therefore$ $p = -$(sum of roots) $= -4$, $q =$ product of roots $= 4 + 3 = 7$

27. **(True)** $f(x) = (x-a)(x-c) + 2(x-b)(x-d)$.

$f(a) = +ve; f(b) = -ve; f(c) = -ve; f(d) = +ve$

$\therefore$ There exists two real and distinct roots one in the interval (a, b) and other in (c, d). True

28. **(False)** $2x^2 + 3x + 1 = 0 \Rightarrow x = -1, -1/2$; both are rationals

$\therefore$ Statement is false.

29. **(b,c,d)** Given that $ax^2 + 2bxy + cy^2 > 0$

and $y, x \in \mathbb{R} - \{0\}$

$\Rightarrow c\left(\frac{y}{x}\right)^2 + 2b\left(\frac{y}{x}\right) + a > 0 \Rightarrow c > 0, D < 0$

$4b^2 - 4ac < 0 \Rightarrow b^2 < ac$

(a) $\left(2, \frac{7}{2}, 6\right)$

$\left(\frac{7}{2}\right)^2 > 2 \times 6$

$\therefore$ Option (a) is incorrect

(b) If $\left(3, b, \frac{1}{12}\right) \in S$

$\Rightarrow b^2 < 3.\frac{1}{12} \Rightarrow b^2 < \frac{1}{4} \Rightarrow 4b^2 < 1$

$\Rightarrow |2b| < 1$ obtion (b) is correct

(c) $ax + by = 1$

$bx + cy = -1$

$D = \begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2 \neq 0$

$\therefore$ unique solution option (c) is correct.

(d) $(a+1)x + by = 0$

$bx + (c+1)y = 0$

$D = \begin{vmatrix} (a+1) & b \\ b & (c+1) \end{vmatrix}$

$= (a+1)(c+1) - b^2 = ac - b^2 + a + c + 1$

Since $ac - b^2 > 0$

$\Rightarrow b^2 < ac \Rightarrow ac$ is + ve

$\Rightarrow a$ and c are positive then $(ac - b^2) + a + c + 1 > 0$

$\therefore$ unique solution

$\therefore$ ooption (d) is correct

30. **(a, b, c)**

$3^x = 4^{x-1} \Rightarrow x\log 3 = 2(x-1)\log 2$

$\Rightarrow x = \frac{2\log 2}{2\log 2 - \log 3}$

$\Rightarrow x = \frac{2\log_3 2}{2\log_3 2 - 1} = \frac{2}{2 - \log_2 3}$

Also $x = \frac{1}{1 - \frac{1}{2}\log_2 3} = \frac{1}{1 - \log_4 3}$

31. **(b)** $\alpha^2 = \alpha + 1$

$\beta^2 = \beta + 1$

$a_n = p\alpha^n + q\beta^n$

$= p(\alpha^{n-1} + \alpha^{n-2}) + q(\beta^{n-1} + \beta^{n-2})$

$= a_{n-1} + a_{n-2}$

$\therefore$ $a_{12} = a_{11} + a_{10}$

32. **(d)** $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$

$a_4 = a_3 + a_2$

$= 2a_2 + a_1$

$= 3a_1 + 2a_0$

$28 = p(3\alpha + 2) + q(3\beta + 2)$

$28 = (p+q)\left(\frac{3}{2} + 2\right) + (p-q)\left(\frac{3\sqrt{5}}{2}\right)$

$\therefore$ $p - q = 0$ and $(p+q) \times \frac{7}{2} = 28$

$\Rightarrow$ $p + q = 8 \Rightarrow$ $p = q = 4$

$\therefore$ $p + 2q = 12$

33. **(b)** As $a, b, c, p, q, \in R$ and the two given equations have exactly one common root

$\Rightarrow$ Either both equations have real roots

or both eqations have imaginary roots

$\Rightarrow$ Either $D_1 \geq 0$ and $D_2 \geq 0$ or $D_1 \leq 0$ and $D_2 \leq 0$

$\Rightarrow$ $p^2 - q \geq 0$ and $b^2 - ac \geq 0$

or $p^2 - q \leq 0$ and $b^2 - ac \leq 0$

$\Rightarrow$ $(p^2 - q)(b^2 - ac) \geq 0$

$\therefore$ Statement 1 is true.

Also we have $\alpha\beta = q$ and $\frac{\alpha}{\beta} = \frac{c}{a}$

$\therefore$ $\frac{\alpha\beta}{\alpha/\beta} = \frac{q}{c} \times a \Rightarrow \beta^2 = \frac{qa}{c}$

As $\beta \neq 1$ or $-1 \Rightarrow \beta^2 \neq 1 \Rightarrow \frac{qa}{c} \neq 1$ or $c \neq qa$

Again, as exactly one root α is common, and $\beta \neq 1$

$\therefore$ $\alpha + \beta \neq \alpha + \frac{1}{\beta} \Rightarrow \frac{-2b}{a} \neq -2p \Rightarrow b \neq ap$

$\therefore$ Statement 2 is correct.

But Statement 2 is not a correct explanation of Statement 1.

34. Roots of $x^2 - 10cx - 11d = 0$ are a and b

$\Rightarrow$ $a + b = 10c$ and $ab = -11d$

Similarly c and d are the roots of $x^2 - 10ax - 11b = 0$

$\Rightarrow$ $c + d = 10a$ and $cd = -11b$

$\Rightarrow$ $a + b + c + d = 10(a + c)$ and $abcd = 121\ bd$

$\Rightarrow$ $b + d = 9(a + c)$ and $ac = 121$

Also we have $a^2 - 10\ ac - 11d = 0$ and $c^2 - 10ac - 11b = 0$

$\Rightarrow$ $a^2 + c^2 - 20ac - 11(b + d) = 0$

$\Rightarrow$ $(a + c)^2 - 22 \times 121 - 99(a + c) = 0$

$\Rightarrow$ $a + c = 121$ or -22

For $a + c = -22$, we get $a = c$

$\therefore$ Rejecting this value we have $a + c = 121$

$\therefore$ $a + b + c + d = 10(a + c) = 1210$

35. Given :

$x^2 + (a - b)x + (1 - a - b) = 0$, $a, b \in R$

For this equation to have unequal real roots for all value of b if $D > 0$

$\Rightarrow$ $(a - b)^2 - 4(1 - a - b) > 0$

$\Rightarrow$ $a^2 + b^2 - 2ab - 4 + 4a + 4b > 0$

$\Rightarrow$ $b^2 + b(4 - 2a) + a^2 + 4a - 4 > 0$

Which is a quadratic expression in b, and it will be true for all $b \in R$, if discriminant of above equation is less than zero.

i.e., $(4 - 2a)^2 - 4(a^2 + 4a - 4) < 0$

$\Rightarrow$ $(2 - a)^2 - (a^2 + 4a - 4) < 0$

$\Rightarrow$ $4 - 4a + a^2 - a^2 - 4a + a < 0$

$\Rightarrow$ $-8a + 8 < 0$, $\therefore$ $a > 1$

36. We know $(\alpha-\beta)^2 = [(\alpha+\delta)-(\beta+\delta)]^2$

$\Rightarrow \quad (\alpha+\beta)^2 - 4\alpha\beta = (\alpha+\delta+\beta+\delta)^2 - 4(\alpha+\delta)(\beta+\delta)$

$\Rightarrow \quad \dfrac{b^2}{a^2} - \dfrac{4c}{a} = \dfrac{B^2}{A^2} - \dfrac{4C}{A} \Rightarrow \dfrac{4ac-b^2}{a^2} = \dfrac{4AC-B^2}{A^2}$

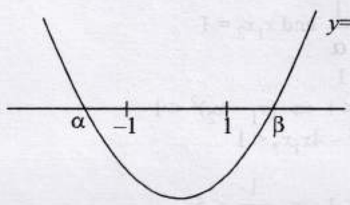
$\left[\text{Here } \alpha+\beta = -\dfrac{b}{a}, \alpha\beta = \dfrac{c}{a}, \right.$

$\left. (\alpha+\delta)\,(\beta+\delta) = -\dfrac{B}{A} \text{ and } (\alpha+\delta)\,(\beta+\delta) = \dfrac{C}{A}\right]$

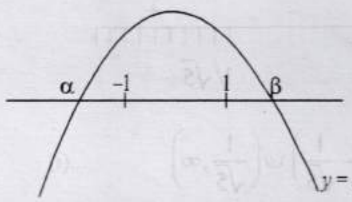
37. Given : For $a, b, c \in R$, $ax^2 + bx + c = 0$ has two real roots α and β, where $\alpha < -1$ and $\beta > 1$. There may be two cases depending upon the value of a, as shown below.

In each of cases (i) and (ii) $af(-1) < 0$ and $af(1) < 0$

(i) If $a > 0$



(ii) If $a < 0$



$\Rightarrow \; a\,(a-b+c) < 0$ and $a\,(a+b+c) < 0$

Dividing by a^2 (> 0), we get

$1 - \dfrac{b}{a} + \dfrac{c}{a} < 0$ ....(i)

and $1 + \dfrac{b}{a} + \dfrac{c}{a} < 0$ ....(ii)

On combining (i) and (ii) we get

$1 + \left|\dfrac{b}{a}\right| + \dfrac{c}{a} < 0$ or $1 + \dfrac{c}{a} + \left|\dfrac{b}{a}\right| < 0$

38. Given :

$|x^2 + 4x + 3| + 2x + 5 = 0$

Here two cases are possible.

Case I : $x^2 + 4x + 3 \geq 0 \Rightarrow (x+1)(x+3) \geq 0$

$\Rightarrow x \in (-\infty, -3] \cup [-1, \infty)$ ....(i)

Then the given equation becomes,

$\Rightarrow \; x^2 + 6x + 8 = 0$

$\Rightarrow \; (x+4)(x+2) = 0, \; \therefore \; x = -4, -2$

But $x = -2$ does not satisfy (i) and hence rejected.

$\therefore$ Solution is $x = -4$

Case II : $x^2 + 4x + 3 < 0$

$\Rightarrow \; (x+1)(x+3) < 0$

$\Rightarrow \; x \in (-3, -1)$ ....(ii)

Then the given equation becomes,

$-(x^2 + 4x + 3) + 2x + 5 = 0$

$\Rightarrow -x^2 - 2x + 2 = 0 \Rightarrow x^2 + 2x - 2 = 0$

$\Rightarrow \; x = \dfrac{-2 \pm \sqrt{4+8}}{2} \; \therefore \; x = -1+\sqrt{3}, -1-\sqrt{3}$

But $x = -1+\sqrt{3}$ does not satisfy (ii) and hence rejected.

$\therefore$ Solution is $x = -1-\sqrt{3}$

On combining solution in the two cases, we get the solutions: $x = -4, -1-\sqrt{3}$.

39. Given :

$x^2 - 2a\,|x-a| - 3a^2 = 0$ .... (i)

Here two cases are possible.

Case I : $x - a > 0$, then $|x-a| = x - a$

Hence, Eq. (i) becomes

$x^2 - 2a\,(x-a) - 3a^2 = 0$

$\Rightarrow \; x^2 - 2ax - a^2 = 0 \Rightarrow x = \dfrac{2a \pm \sqrt{4a^2 + 4a^2}}{2}$

$\therefore \; x = a \pm a\sqrt{2}$

Case II : $x - a < 0$, then $|x-a| = -(x-a)$

Hence, Eq. (i) becomes

$x^2 + 2a\,(x-a) - 3a^2 = 0$

$\Rightarrow \; x^2 + 2ax - 5a^2 = 0 \Rightarrow \quad x = \dfrac{-2a \pm \sqrt{4a^2 + 20a^2}}{2}$

$\therefore \quad x = \dfrac{-2a \pm 2a\sqrt{6}}{2} \Rightarrow \quad x = -a \pm a\sqrt{6}$

Hence, the solution set is $\{a \pm a\sqrt{2}, -a \pm a\sqrt{6}\}$

40. Given, $(5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10$ ....(i)

Put $y = (5+2\sqrt{6})^{x^2-3} \Rightarrow (5-2\sqrt{6})^{x^2-3} = \dfrac{1}{y}$

From Eq. (i), $y + \dfrac{1}{y} = 10$

$\Rightarrow \; y^2 - 10y + 1 = 0 \Rightarrow y = 5 \pm 2\sqrt{6}$

$\Rightarrow \; (5+2\sqrt{6})^{x^2-3} = 5+2\sqrt{6}$

or $\; (5+2\sqrt{6})^{x^2-3} = 5-2\sqrt{6}$

$\Rightarrow \; x^2 - 3 = 1$ or $x^2 - 3 = -1$

$\Rightarrow x = \pm 2$ or $x = \pm\sqrt{2} \Rightarrow x = \pm 2, \pm\sqrt{2}$

41. Given $a > 0$, so we have to consider two cases :

$a \neq 1$ and $a = 1$.

Also it is clear that $x > 0$

and $x \neq 1$, $ax \neq 1$, $a^2x \neq 1$.

Case I : If $a > 0, \neq 1$

then given equation can be simplified as

$$\frac{2}{\log_a x} + \frac{1}{1+\log_a x} + \frac{3}{2+\log_a x} = 0$$

Putting $\log_a x = y$, we get

$2(1+y)(2+y) + y(2+y) + 3y(1+y) = 0$

$\Rightarrow \quad 6y^2 + 11y + 4 = 0 \Rightarrow y = -4/3$ and $-1/2$

$\Rightarrow \quad \log_a x = -4/3$ and $\log_a x = -1/2$

$\Rightarrow \quad x = a^{-4/3}$ and $x = a^{-1/2}$

Case II : If $a = 1$, then equation becomes

$2\log_x 1 + \log_x 1 + 3\log_x 1 = 6\log_x 1 = 0$

which is true $\forall x > 0, \neq 1$

Hence solution is $x > 0, \neq 1$; if $a = 1$,

and $x = a^{-1/2}, a^{-4/3}$, if $a > 0, \neq 1$

42. $\sqrt{x+1} = 1 + \sqrt{x-1}$

Squaring both sides, we get

$x + 1 = 1 + x - 1 + 2\sqrt{x-1} \Rightarrow 1 = 2\sqrt{x-1}$.

$\Rightarrow \quad 1 = 4(x-1) \Rightarrow x = 5/4$

Topic-4: Condition for Common Roots, Maximum and Minimum value of Quadratic Equation, Quadratic Expression in two Variables, Solution of Quadratic Inequalities

1. **(b)** Let α be the common root of given equations, then

$\alpha^2 + b\alpha - 1 = 0$ ...(i)

and $\alpha^2 + \alpha + b = 0$ ...(ii)

On subtracting (ii) from (i), we get

$(b-1)\alpha - (b+1) = 0$

$\Rightarrow \alpha = \dfrac{b+1}{b-1}$

Substituting this value of α in equation (i), we get

$$\left(\frac{b+1}{b-1}\right)^2 + b\left(\frac{b+1}{b-1}\right) - 1 = 0 \Rightarrow b^3 + 3b = 0$$

$\Rightarrow b = 0, i\sqrt{3}, -i\sqrt{3}$

2. **(b)** $f(x) = ax^2 + bx + c$ has same sign as that of a if $D < 0$.

Since $x^2 + 2ax + 10 - 3a > 0 \; \forall x$

$\therefore \quad D < 0 \Rightarrow 4a^2 - 4(10-3a) < 0 \Rightarrow a^2 + 3a - 10 < 0$

$\Rightarrow \quad (a+5)(a-2) < 0 \Rightarrow a \in (-5, 2)$

3. **(20)** Given that $f(1) = -9 \Rightarrow 1 + a + b + c = -9$ ...(i)

and $4x^3 + 3ax^2 + 2bx = 0$

$\Rightarrow x = 0$, or $4x^2 + 3ax + 2b = 0$ ...(ii)

$\Rightarrow \sqrt{3}i$ and $-\sqrt{3}i$ are roots of (ii)

$\Rightarrow \sqrt{3}i - \sqrt{3}i = \dfrac{-3a}{4}, \sqrt{3}i(-\sqrt{3}i) = \dfrac{2b}{4}$

$\Rightarrow a = 0, b = 6, c = -16$ from (i)

$\Rightarrow f(x) = 0 \Rightarrow x^4 + 6x^2 - 16 = 0$

$\Rightarrow x^2 = \dfrac{-6 \pm \sqrt{36+64}}{2} = -3 \pm 5 = 2, -8$

$x = -\sqrt{2}, +\sqrt{2}, -2\sqrt{2}i, 2\sqrt{2}i$

$\Rightarrow |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 = 20$

4. $\because$ x = 1, reduces both the equations to $1 + a + b = 0$

$\therefore$ 1 is the common root. for $a + b = -1$

$\therefore$ Numerical value of $a + b = 1$

5. **(True)** $P(x).Q(x) = (ax^2 + bx + c)(-ax^2 + bx + c)$

$\Rightarrow D_1 = b^2 - 4ac$ and $D_2 = b^2 + 4ac$

clearly, $D_1 + D_2 = 2b^2 \geq 0$

$\therefore$ Atleast one of D_1 and D_2 is positive. Hence, atleast two real roots. True

6. **(a, d)** Given, x_1 and x_2 are roots of $\alpha x^2 - x + \alpha = 0$.

$\therefore \quad x_1 + x_2 = \dfrac{1}{\alpha}$ and $x_1 x_2 = 1$

Also, $|x_1 - x_2| < 1$

$\Rightarrow \quad |x_1 - x_2|^2 < 1 \Rightarrow (x_1 - x_2)^2 < 1$

or $\quad (x_1 + x_2)^2 - 4x_1x_2 < 1$

$\Rightarrow \quad \dfrac{1}{\alpha^2} - 4 < 1$ or $\dfrac{1}{\alpha^2} < 5$

or $\quad 5\alpha^2 - 1 > 0$ or $(\sqrt{5}\alpha - 1)(\sqrt{5}\alpha + 1) > 0$

$\therefore \quad \alpha \in \left(-\infty, -\dfrac{1}{\sqrt{5}}\right) \cup \left(\dfrac{1}{\sqrt{5}}, \infty\right)$ ...(i)

Also, $D > 0$

$\Rightarrow \quad 1 - 4\alpha^2 > 0$ or $\alpha \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ ...(ii)

From Eqs. (i) and (ii), we get

$$\alpha \in \left(-\frac{1}{2}, \frac{-1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$$

7. **(b)** Given : a, b, c, d, p are real and distinct numbers such that

$(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \leq 0$

$\Rightarrow \quad (a^2p^2 + b^2p^2 + c^2p^2) - (2abp + 2bcp + 2cdp)$
$+ (b^2 + c^2 + d^2) \leq 0$

$\Rightarrow \quad (a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2)$
$+ (c^2p^2 - 2cdp + d^2) \leq 0$

$\Rightarrow \quad (ap - b)^2 + (bp - c)^2 + (cp - d)^2 \leq 0$

Since, LHS is the sum of perfect squares, therefore LHS can never be –ve.

$\therefore (ap - b)^2 + (bp - c)^2 + (cp - d)^2 = 0$

Which is possible only when each term is zero individually i.e. $ap-b=0;\ bp-c=0;\ cp-d=0$

$\Rightarrow \frac{b}{a}=p; \frac{c}{b}=p; \frac{d}{c}=p \Rightarrow \frac{b}{a}=\frac{c}{b}=\frac{d}{c}=p$

$\therefore$ a,b,c,d are in G.P.

8. **(c, d)** Let $y=\frac{(x-a)(x-b)}{(x-c)}$

$\Rightarrow (x-c)y=x^2-(a+b)x+ab$

$\Rightarrow x^2-(a+b+y)x+ab+cy=0$

Here, $D=(a+b+y)^2-4(ab+cy)$

$=y^2+2y(a+b-2c)+(a-b)^2$

Since x is real and y assumes all real values.

$\therefore$ $D \geq 0$ for all real values of y

$\Rightarrow y^2+2y(a+b-2c)+(a-b)^2 \geq 0$

As we know that the sign of a quadratic polynomial is same as that of coefficient of y^2 if its descriminant < 0

$\therefore$ $4(a+b-2c)^2-4(a-b)^2<0$

$\Rightarrow 4(a+b-2c+a-b)(a+b-2c-a+b)<0$

$\Rightarrow 16\,(a-c)\,(b-c)<0$

$\Rightarrow 16\,(c-a)\,(c-b)<0$ ....(i)

If $a<b$ then from inequation (i), we get $c \in (a, b)$

$\Rightarrow a<c<b$

If $a>b$ then from inequation (i), we get $c \in (b, a)$

$\Rightarrow a>c>b$

Thus, both (c) and (d) are the correct answer.

9. Given : $ax^2+bx+c=0$ .... (i)

and $a^3x^2+abcx+c^2=0$ .... (ii)

$\therefore$ $\alpha+\beta=-\frac{b}{a},\ \alpha.\beta=\frac{c}{a}$

Divide the equation (ii) by a^3, we get

$x^2+\frac{b}{a}.\frac{c}{a}.x+\left(\frac{c}{a}\right)^3=0$

$\Rightarrow x^2-(\alpha+\beta).(\alpha\beta)\,x+(\alpha\beta)^3=0$

$\Rightarrow x^2-\alpha^2\beta x-\alpha\beta^2 x+(\alpha\beta)^3=0$

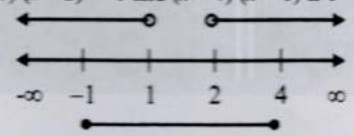
$\Rightarrow x\,(x-\alpha^2\beta)-\alpha\beta^2\,(x-\alpha^2\beta)=0$

$\Rightarrow (x-\alpha^2\beta)\,(x-\alpha\beta^2)=0$

$\Rightarrow x=\alpha^2\beta,\ \alpha\beta^2$

10. Given : $x^2-3x+2>0,\quad x^2-3x-4\leq 0$

$\Rightarrow (x-1)\,(x-2)>0$ and $(x-4)\,(x+1)\leq 0$



$\Rightarrow x \in (-\infty, 1) \cup (2, \infty)$ and $x \in [-1, 4]$

$\therefore$ Common solution $=[-1, 1) \cup (2, 4]$

11. $\because$ α, β are the roots of $x^2+px+q=0$

$\therefore$ $\alpha+\beta=-p,\ \alpha\beta=q$

$\because$ γ, δ are the roots of $x^2+rx+s=0$

$\therefore$ $\gamma+\delta=-r, \gamma\delta=s$

Now, $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$

$=[\alpha^2-(\gamma+\delta)\alpha+\gamma\delta][\beta^2-(\gamma+\delta)\beta+\gamma\delta]$

$=[\alpha^2+r\alpha+s][\beta^2+r\beta+s]$

$\because$ α, β are roots of $x^2+px+q=0$

$\therefore$ $\alpha^2+p\alpha+q=0$ and $\beta^2+p\beta+q=0$

$\Rightarrow$ $\alpha^2=-p\alpha-q$ and $\beta^2=-p\beta-q$

$\therefore (\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)=[\alpha^2+r\alpha+s][\beta^2+r\beta+s]$

$=[(r-p)\alpha+(s-q)][(r-p)\beta+(s-q)]$

$=(r-p)^2\alpha\beta+(r-p)(s-q)(\alpha+\beta)+(s-q)^2$

$=q(r-p)^2-p(r-p)(s-q)+(s-q)^2$

Now if the equations $x^2+px+q=0$ and $x^2+rx+s=0$ have a common root say α, then $\alpha^2+p\alpha+q=0$ and $\alpha^2+r\alpha+s=0$

$\Rightarrow \frac{\alpha^2}{ps-qr}=\frac{\alpha}{q-s}=\frac{1}{r-p}$

$\Rightarrow \alpha^2=\frac{ps-qr}{r-p}$ and $\alpha=\frac{q-s}{r-p}$

$\Rightarrow (q-s)^2=(r-p)(ps-qr)$, which is the required condition.